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STRESSES AND STRAINS AROUND A CYLINDRICAL TUNNEL IN AN ELASTO-PLASTIC MATERIAL WITH DILATANCY

by

ALFRED J. HENDRON, JR. AND A. K. AIYER

SEPTEMBER 1972

OMAHA DISTRICT, CORPS OF ENGINEERS
OMAHA, NEBRASKA 68102

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13. ABSTRACT

In ~~this~~ report an elasto-plastic solution is given for determining the stresses, strains, and displacements around a circular tunnel in rock. The relationship between the principal stresses in the plastic zone are determined by the Coulomb-Navier failure criteria and the concept of "normality" is used to relate the plastic strains to the "yield' surface. The "normality" concept results in a dilatancy of the material at yielding when the angle of shearing resistance of the material is greater than zero. Thus the displacements calculated by the method given in this report include the effect of dilatancy in the yielded zone around the tunnel.

The results of the study are presented in graphical form which enable a wide variety of problems to be solved by hand without the use of a digital computer. Example problems are given in which the method is applied to various situations including lined tunnels in a rock mass which contains a destressed or loosened zone around the tunnel before installation of the liner. The method of analysis presented is especially applicable to problems where loads are imposed after the liner is installed as in the case of the design of protective structures. The method given herein is useful for design because the calculation of deformations and strains, rather than stresses, enable the designer to determine if a particular type of liner can tolerate the deformations imposed at the rock-liner interface.

Suggestions are also given for the selection of rock mass properties to be used in the analysis, i.e. deformation moduli and shear strength properties.

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Rock Strength						
Rock Stress						
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Tunnels						

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PREFACE

This investigation was authorized by the Chief of Engineers (ENGMC-EM) and was performed in FY 1971 under Contract DACA 45-69-C-0100, between the Omaha District, Corps of Engineers and Dr. Alfred J. Hendron, Jr., Foundation Engineer, Mahomet, Illinois. This work is a part of a continuing effort to develop methods which can be used to design underground openings in jointed rock to survive the effects of nuclear weapons.

The report was prepared by Dr. Alfred J. Hendron, Jr. A. K. Aiyer contributed to the report.

During the work period covered by this report, Colonel B. P. Pendergrass was District Engineer; Charles L. Hipp and R. G. Burnett were Chiefs, Engineering Division; C. J. Distefano was Technical Monitor for the Omaha District under general supervision of Kendall C. Fox, Chief, Protective Structures Branch. Dr. J. D. Smart and D. G. Heitmann participated in the monitoring work.

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FIGURES

NOTATION

a = Radius of a circular tunnel or tunnel lining

C = Constant of integration

D = Tunnel diameter

E = Young's modulus of an elastic medium under plane strain conditions

E' = Young's modulus of a loosened or destressed zone around a tunnel

E_o = Young's modulus for an elastic material under plane stress conditions

E_r = Effective Young's modulus of deformation for a rock mass

E_s = Young's modulus for steel

E_{seis} = Dynamic value of Young's modulus calculated from seismic surveys

h_s = Thickness of steel lining

N_ϕ = $\frac{1 + \sin \phi}{1 - \sin \phi}$

P = Difference between the circumferential and radial stresses in the plastic zone, $(\sigma_\theta - \sigma_r)$

p_i = Radial pressure on the surface of a rock tunnel

p_o = Uniform free-field stress

Q = $p_o + T$

q_u = Unconfined compressive strength

R = Radius to the elastic-plastic boundary

R_1 = Radius of loosened or destressed zone

R_2 = Radius of elastic-plastic boundary in the loosened medium, if any, when $R > R_1$

r = Radial distance from the center of a tunnel

S = Stress difference in the elastic region $(\sigma_\theta - \sigma_r)$

s = Joint spacing

T = $\frac{\sigma_u}{N_\phi - 1}$

u = Radial displacement

ϵ_θ = Tangential strain

$\epsilon_{\theta p}$ = Tangential plastic strain

ϵ_r = Radial strain

ϵ_{rp} = Radial plastic strain

μ = Poisson's ratio of medium in plane strain

μ' = Poisson's ratio of loosened or destressed meiium

μ_o = Poisson's ratic for plane stress

ϕ = Angle of shearing resistance of the material

σ_r = Radial stress

σ_{ra} = Radial pressure exerted by steel liner against the adjacent rock medium

σ_θ = Circumferential stress or tangential stresses

σ_R = Radial stress at radius R

σ_u = Unconfined compressive strength of the rock mass surrounding a tunnel

σ_y = Yield stress of steel

$\tau_{r\theta}$ = Shear stress in polar co-ordinates

CONVERSION FACTORS, BRITISH TO METRIC UNITS OF MEASUREMENT

British units of measurement used in this report can be converted to metric units as follows:

Multiply	By	To Obtain
inches	2.54	centimeters
feet	0.3048	meters
cubic inches	16.3871	cubic centimeters
pounds	0.45359237	kilograms
pounds per square inch	0.070307	kilograms per square centimeter
pounds per cubic foot	16.0185	kilograms per cubic meter
inch-pounds	0.011521	meter-kilograms
inches per second	2.54	centimeters per second

Chapter 1

Introduction

General

One of the most significant engineering properties of rock masses is shear strength. The stability of both lined and unlined openings under various static and dynamic loadings depends on the shear strength of the surrounding mass. Since the shear strength of rock masses increases with confining pressure, it is necessary to employ an analysis which includes this property in order to properly assess the effects of various types of tunnel linings.

One of the first attempts to take into account the Coulomb-Navier shear strength characteristics of earth materials in a yielded zone around a deep tunnel was made by Terzaghi (1919). Although he did not obtain a general solution to this problem he was impressed by his field observation that deep boreholes in frictional materials remained stable at depths where the material adjacent to the opening should have failed. His curiosity about this problem led him to ask H. M. Westergaard, a colleague of his at Harvard, two questions.

(1) "What distributions of stress are possible in the soil around an unlined drill hole for a deep well?"

(2) "What distributions of stress make it possible for the hole not to collapse but remain stable for some time either with no lining or with a thin "stove-pipe" lining of small structural strength?"

Westergaard (1940) published the answers to these questions after obtaining a solution for the stresses around a borehole where the material in the plastic zone around the hole was assumed to fail according to the Coulomb-Navier

failure criteria. The results showed that a small radial confining pressure at the surface of the borehole enabled the radial stresses to increase rapidly with depth behind the surface such that the radial pressures a few inches behind the borehole surface furnished sufficient confinement to support the high circumferential stresses around the opening. The results of Westergaard's solution were expanded and interpreted by Terzaghi, 1943. Essentially the same solution has been published by Jaeger (1956), Jaeger and Cook (1969) and Sirieys (1964). In this solution, the stresses are obtained around a long circular tunnel in a medium where the free-field principal stresses are equal (i.e. $\sigma_v = \sigma_H = p_o$, Fig. 1). For this case, the shear stresses around the opening, $\tau_{r\theta}$, are equal to zero and the differential equation of equilibrium shown in Fig. 1 can be expressed in terms of the radial stress σ_r and circumferential stress σ_θ. The solution was obtained by integrating the differential equation of equilibrium in the plastic zone (Fig. 1) with the constraint that the circumferential and radial stresses are related by the Coulomb-Navier failure criteria and requiring that the radial stresses are continuous at the boundary between the elastic and plastic zones (Fig. 1).

An illustration of the stress distributions obtained from the Westergaard solution is given in Fig. 2 for a tunnel in sand ($C = 0$, $\phi = 30°$). The distributions of circumferential stress σ_θ and radial stress σ_r are shown in dimensionless form in terms of the free-field stress p_o and three different stress distributions are shown for cases where the confining pressure on the inside of the tunnel is equal to 1/10, 1/20, and 1/40 of the free-field stress p_o. For each stress distribution shown the circumferential stress is maximum at the boundary of the elastic and plastic zones and the distance to the boundary between the elastic and plastic zones increases as the ratio of the free-field stress, p_o, to the internal pressure, p_i, increases. It should also be noted that the circumferential stress increases very rapidly with depth behind the opening in the plastic zone

and illustrates the ability of a frictional material to carry high circumferential stresses at depth due to providing a nominal confining pressure, p_i, on the inside surface of the tunnel.

The solution discussed above however yields only the elastic-plastic stress distribution around the opening. A knowledge of the stress distribution is only of academic interest for the design of a tunnel lining unless the deformations at the wall of the tunnel are compatible with the deformations a lining can resist before failing. The radial displacements at a tunnel wall are due to both the inelastic strains in the plastic zone adjacent to the tunnel and the elastic strains in the rock outside the plastic zone. Although the strains in the elastic region are easily calculated, the calculation of strains in the plastic zone involves some assumption regarding the relations between the plastic strains and volume changes of the material at failure. Newmark (1969) obtained a closed solution for the problem described above and shown in Fig. 1 which considers the displacements and strains around the opening as well as the stress distributions. To obtain this solution Newmark combined the differential equation of equilibrium and the compatibility equation relating radial displacements to radial strains and circumferential strains to yield the relation given as equation (4) in this report. Since the differential equation of equilibrium and the compatibility relation involve only equilibrium and geometry respectively it follows that the relation given by equation (4) must be satisfied in both the elastic and plastic regions. Newmark (1969) obtained a solution for equation (4) by assuming that after yield of the material that the plastic components of the radial and circumferential strains were such that the resulting volume change was zero or the yielded material behaved as an incompressible solid. The results of the solution have been prepared by Newmark (1969) in the form of charts from which the displacements at the tunnel wall can be determined as a function of the elastic properties of the

medium (ν, E), the shear strength properties of the medium as determined by the unconfined compressive strength and the angle of internal friction, the free-field stress, and the capacity of the tunnel lining. Since real rock masses increase in volume at failure, a phenomenon called "dilatancy", it is probable that the Newmark solution underestimates the inward radial displacements at the tunnel wall because of the assumption that the plastic components of strain yield no volume change.

Scope

In this report an elasto-plastic solution is given in Chapter 2 which accounts for the dilatant properties of a rock mass which obeys the Coulomb-Navier failure criterion. Example calculations are given to illustrate the variety of problems which may be solved by this method. Suggestions for the selection of rock mass properties and conclusions are given in Chapter 3.

Chapter 2

Analysis

The basic configuration considered in this study is shown in Fig. 1. This illustration represents a section through an infinitely long tunnel where the strain in the direction of the tunnel axis is zero. The cylindrical coordinate system with the origin at the center of the circular opening is used in this analysis.

Fundamental Relations of Equilibrium and Compatibility

The relations given herein are derived for plane strain conditions for a uniform stress field.

The basic differential equation of equilibrium for a typical element shown in Fig. 1(a) is

$$\frac{\partial \sigma_r}{\partial r} + \frac{1}{r} (\sigma_r - \sigma_\theta) = 0 \qquad (1)$$

where σ_r and σ_θ represent the radial and tangential stresses respectively, at a radial distance r.

The radial and tangential strains for the assumed conditions of plane strain can be stated in terms of the radial displacement, u, as follows:

$$\varepsilon_r = \frac{\partial u}{\partial r} \qquad \varepsilon_\theta = \frac{u}{r} \qquad (2)$$

Upon eliminating the displacement u from Eqs. (2), one obtains the following equation of compatibility:

$$r \frac{\partial \varepsilon_\theta}{\partial r} + (\varepsilon_\theta - \varepsilon_r) = 0 \qquad (3)$$

Eqs. (1) and (3) can now be combined into a single equation by multiplying Eq. (1) by the quantity $-(1-\mu)r$ and adding it to Eq. (3) multiplied by the quantity E. The result is:

$$r\frac{\partial}{\partial r}\left[E\varepsilon_\theta-(1-\mu)\sigma_r\right] + \left[E\varepsilon_\theta-(1-\mu)\sigma_r\right] - \left[E\varepsilon_r-(1-\mu)\sigma_\theta\right] = 0 \qquad (4)$$

It is to be noted here that Eqs. (1), (3) and (4) are valid for both elastic and inelastic conditions. However, the validity of Eqs. (2), (3) and (4) is limited to the case of small displacements.

In order to solve Eq. (4), it is necessary to relate the term in brackets at the extreme right of Eq. (4) to the other terms in that equation through the use of stress-strain relationships and to establish the appropriate boundary conditions.

Relations for the Elastic Zone

The stress-strain relationships for the elastic behavior of a material under plane strain conditions can be written as follows:

$$E\varepsilon_\theta = \sigma_\theta - \mu\sigma_r \qquad (5)$$

$$E\varepsilon_r = \sigma_r - \mu\sigma_\theta \qquad (6)$$

where E and μ are the Young's modulus and Poisson's ratio of the material under plane strain conditions. The values of E and μ for plane strain are related to the corresponding values for plane stress by the following equations:

$$E = E_o/(1-\mu_o^2) \qquad (7)$$

$$\mu = \mu_o/(1-\mu_o) \qquad (8)$$

where E_o is the Young's modulus for plane stress and μ_o is the Poisson's ratio for plane stress. If the quantity S, as suggested by Newmark (1969), is defined by,

$$S = E\varepsilon_\theta - (1 - \mu)\,\sigma_r \tag{9}$$

then from Eq. (5) it follows that

$$S = (\sigma_\theta - \sigma_r) \tag{10}$$

Thus in the elastic region the quantity S is equal to the stress difference between the tangential stress σ_θ and the radial stress σ_r. Eq. (6) gives

$$E\varepsilon_r - (1 - \mu)\,\sigma_\theta = \sigma_r - \sigma_\theta = -S \tag{11}$$

Using Eqs. (9) and (11), Eq. (4) can be simplified as:

$$r\,\frac{\partial S}{\partial r} + 2S = 0 \tag{12}$$

or $$r\,\frac{\partial}{\partial r}\,(\sigma_\theta - \sigma_r) + 2\,(\sigma_\theta - \sigma_r) = 0 \tag{13}$$

From Eq. (1) $$2r\,\frac{\partial \sigma_r}{\partial r} + 2\,(\sigma_r - \sigma_\theta) = 0 \tag{14}$$

Adding Eqs. (13) and (14),

$$r\,\frac{\partial}{\partial r}\,(\sigma_\theta + \sigma_r) = 0 \tag{15}$$

Eq. (15) indicates that in the elastic region, the sum $(\sigma_\theta + \sigma_r)$ is a constant. At very large values of r, each of the stresses σ_θ and σ_r is equal to the free-field uniform stress, p_o. Therefore in the elastic region,

$$\sigma_\theta + \sigma_r = 2p_o \tag{16}$$

and
$$S = \sigma_\theta - \sigma_r = 2p_o - 2\sigma_r \tag{16a}$$

Integration of Eq. (12) yields

$$S = S_R \left(\frac{R}{r}\right)^2 \tag{17}$$

where S_R represents the value of S at radius R in the elastic region,

i.e.,
$$S_R = S\Big|_{r=R} = 2p_o - 2\sigma_R \tag{18}$$

where σ_R = radial stress at radius R in the elastic zone.

At any radius r in the elastic zone,

$$\sigma_r = \frac{2p_o - S}{2} = p_o - (p_o - \sigma_R) \left(\frac{R}{r}\right)^2 \tag{19}$$

$$\sigma_\theta = 2p_o - \sigma_r = p_o + (p_o - \sigma_R) \left(\frac{R}{r}\right)^2 \tag{20}$$

<u>Relations for the Plastic Zone</u>

The radial and circumferential (or tangential) strains in the plastic region can be written as the sum of the respective elastic and plastic components. Thus in the plastic zone,

$$\begin{aligned} \varepsilon_\theta &= \varepsilon_\theta \text{ (elastic)} + \varepsilon_\theta \text{ (plastic)} \\ \text{and} \quad \varepsilon_r &= \varepsilon_r \text{ (elastic)} + \varepsilon_r \text{ (plastic)} \end{aligned} \tag{21}$$

The concept of perfect plasticity requires that the associated strain rate vector must be normal to the yield surface, Drucker and Prager (1953). Accordingly, if f represents the yield function which is valid for the plastic zone, the plastic strain rate components in the radial and tangential directions can be related by the equation

$$\frac{\dot{\varepsilon}_\theta \text{ (plastic)}}{\dot{\varepsilon}_r \text{ (plastic)}} = \frac{\partial f/\partial \sigma_\theta}{\partial f/\partial \sigma_r} \qquad (22)$$

The yield function used in this analysis is based on Coulomb-Navier yield criterion and is given by

$$f = \sigma_\theta - \sigma_r \cdot N_\phi - \sigma_u = 0 \qquad (23)$$

where $N_\phi = \frac{1 + \sin\phi}{1 - \sin\phi}$, σ_u = unconfined compressive strength of the material and ϕ = angle of shearing resistance of the material. Thus Eq. (22) gives

$$\frac{\dot{\varepsilon}_\theta \text{ (plastic)}}{\dot{\varepsilon}_r \text{ (plastic)}} = -\frac{1}{N_\phi} = \text{constant.}$$

Since the ratio of the plastic strain rate components is a constant (for the yield function used in this analysis), the ratio of the plastic strain components will also be equal to the same constant. Thus the ratio between the plastic strain components may be written as

$$\frac{\varepsilon_\theta \text{ (plastic)}}{\varepsilon_r \text{ (plastic)}} = -\frac{1}{N_\phi}$$

or $$\varepsilon_\theta \text{ (plastic)} = \varepsilon_{\theta p} = -\frac{1}{N_\phi} \cdot \varepsilon_r \text{ (plastic)}$$

or $$\varepsilon_r \text{ (plastic)} = -N_\phi \cdot \varepsilon_{\theta p} \qquad (24)$$

It should be noted that the relation given in Eq. (24) produces an increase in volume at failure and the percentage of volume change increases as N_ϕ increases. Eq. (24) accounts for the primary difference between the solution presented herein and the solution given by Newmark (1969) where the condition of incompressibility assumed by Newmark imposed the condition that $\epsilon_{rp} + \epsilon_{\theta p} = 0$. By utilizing Eq. (24) Eqs. (21) can now be rewritten as:

$$\epsilon_\theta = \frac{1}{E}(\sigma_\theta - \mu\sigma_r) + \epsilon_{\theta p} \tag{25}$$

$$\epsilon_r = \frac{1}{E}(\sigma_r - \mu\sigma_\theta) - N_\phi \cdot \epsilon_{\theta p} \tag{26}$$

or

$$E\epsilon_\theta = \sigma_\theta - \mu\sigma_r + E \cdot \epsilon_{\theta p} \tag{27}$$

$$E\epsilon_r = \sigma_r - \mu\sigma_\theta - N_\phi \cdot E \cdot \epsilon_{\theta p} \tag{28}$$

Substituting Eqs. (27) and (28) into Eq. (4)

$$r\frac{\partial S}{\partial r} + 2S + E\ \epsilon_{\theta p}(N_\phi - 1) = 0 \tag{29}$$

where $S = E\epsilon_\theta - (1 - \mu)\,\sigma_r$

The quantity S is equal to the difference between the tangential and radial stresses in the elastic range but in the inelastic (plastic) range it has only a formal meaning. For the inelastic conditions, the difference between the tangential stress σ_θ and the radial stress σ_r is defined as:

$$P = \sigma_\theta - \sigma_r \tag{30}$$

By Eq. (27), $$E\epsilon_{\theta p} = E\epsilon_\theta - \sigma_\theta + \mu\sigma_r = S - (\sigma_\theta - \sigma_r) = S - P \tag{31}$$

Therefore Eq. (29) becomes

$$r \frac{\partial S}{\partial r} + 2S + (S - P)\ (N_\phi - 1) = 0$$

or

$$r \frac{\partial S}{\partial r} + S\ (N_\phi + 1) = P\ (N_\phi - 1) \tag{32}$$

It may be noted that Eq. (32) is valid in the plastic region only.

In the plastic region the Coulomb-Navier yield criterion is assumed to be valid. Accordingly

$$\sigma_\theta = \sigma_r \cdot N_\phi + \sigma_u \tag{33}$$

$$P = \sigma_\theta - \sigma_r = (N_\phi - 1)\ \sigma_r + \sigma_u \tag{34}$$

Differentiating Eq. (34) with respect to r

$$\frac{\partial P}{\partial r} = (N_\phi - 1) \frac{\partial \sigma_r}{\partial r} \tag{35}$$

Substituting Eq. (35) into Eq. (1)

$$r \frac{\partial P}{\partial r} - (N_\phi - 1)\ P = 0 \tag{36}$$

The solution of Eq. (36) yields

$$P = P_a \left(\frac{r}{a}\right)^{N_\phi - 1} \tag{37}$$

where $P_a = P\big|_{r=a}$

Let the radial pressure inside the tunnel be p_i and the radius of the circular tunnel be a. Then

$$P_a = P\Big|_{r=a} = \sigma_u + (N_\phi - 1)\, p_i \tag{38}$$

Therefore
$$P = P_a \left(\frac{r}{a}\right)^{N_\phi - 1} = \left[\sigma_u + (N_\phi - 1)\, p_i\right] \left(\frac{r}{a}\right)^{N_\phi - 1} \tag{39}$$

At any radius r within the plastic zone,

$$\sigma_r = (P - \sigma_u)/(N_\phi - 1) = \left(p_i + \frac{\sigma_u}{N_\phi - 1}\right)\left(\frac{r}{a}\right)^{N_\phi - 1} - \frac{\sigma_u}{N_\phi - 1}$$

$$= (p_i + T)\left(\frac{r}{a}\right)^{N_\phi - 1} - T \tag{40}$$

where
$$T = \sigma_u/(N_\phi - 1) \quad \text{and} \tag{41}$$

$$\sigma_\theta = N_\phi \cdot \sigma_r + \sigma_u \tag{33}$$

Substitution of Eq. (39) into Eq. (32) gives

$$r\frac{\partial S}{\partial r} + S\,(N_\phi + 1) = (N_\phi - 1)\left[\sigma_u + p_i\,(N_\phi - 1)\right]\left(\frac{r}{a}\right)^{N_\phi - 1}$$

$$= A\, r^{N_\phi - 1} \tag{42}$$

where

$$A = \frac{(N_\phi - 1)^2\,(p_i + T)}{a^{N_\phi - 1}} \tag{43}$$

Solution of Eq. (42) is of the form

$$S\, r^{N_\phi + 1} = \left(\frac{A}{2N_\phi}\right) r^{2N_\phi} + C \tag{44}$$

where C is the constant of integration. The constant C has to be evaluated from the boundary conditions specified for any given case.

Seven specific cases have been studied in this investigation, each with a different set of boundary conditions. These cases are presented and analyzed in the remainder of this chapter.

Case 1. <u>Tunnel of radius a, supported with a constant internal pressure p_i -- E, μ, σ_u, and ϕ for the plastic zone are the same as for the elastic zone.</u> (Fig. 1b)

Let the radius of the elasto-plastic boundary be R. Let the radial and tangential stresses at r = R be represented by σ_R and $\sigma_{\theta R}$ respectively. According to Eq. (39)

$$P_R = P\Big|_{r=R} = \left[\sigma_u + (N_\phi - 1)\, p_i\right] \left(\frac{R}{a}\right)^{N_\phi - 1}$$

$$= (N_\phi - 1)\,(p_i + T) \left(\frac{R}{a}\right)^{N_\phi - 1} \qquad (45)$$

From Eqs. (33) and (20)

$$\sigma_{\theta R} = \sigma_R \cdot N_\phi + \sigma_u = 2p_o - \sigma_R$$

$$\therefore \quad \sigma_R = \frac{2p_o - \sigma_u}{N_\phi + 1} \qquad (46)$$

Substituting Eq. (46) into Eq. (34)

$$P_R = \sigma_u + (N_\phi - 1)\,\sigma_R = \sigma_u + \frac{N_\phi - 1}{N_\phi + 1} \cdot (2p_o - \sigma_u)$$

$$= \frac{2}{N_\phi + 1}\,(p_o + T)\,(N_\phi - 1) \tag{47}$$

Equating Eqs. (45) and (47)

$$\left(\frac{R}{a}\right)^{N_\phi - 1} = \frac{2}{N_\phi + 1} \cdot \frac{p_o + T}{p_i + T}$$

or

$$\frac{R}{a} = \left(\frac{2}{N_\phi + 1} \cdot \frac{p_o + T}{p_i + T}\right)^{\frac{1}{N_\phi - 1}} \tag{48}$$

At $r = R$,

$$S_R = P_R = 2\,(p_o + T)\,\frac{(N_\phi - 1)}{(N_\phi + 1)} \tag{49}$$

Now the value of the constant in Eq. (44) can be estimated.

According to Eq. (44), at $r = R$

$$C = S_R \cdot R^{N_\phi + 1} - \frac{A \cdot R^{2N_\phi}}{2N_\phi} \tag{50}$$

Substitution of Eq. (50) into Eq. (44)

$$Sr^{N_\phi + 1} = \frac{A}{2N_\phi}\left[r^{2N_\phi} - R^{2N_\phi}\right] + S_R \cdot R^{N_\phi + 1} \tag{51}$$

Substituting for A and S_R in Eq. (51), and simplifying

$$\left(\frac{S}{p_i + T}\right)\left(\frac{r}{a}\right)^{N_\phi + 1} = \frac{N_\phi - 1}{2N_\phi}\left[\left(\frac{r}{a}\right)^{2N_\phi} \cdot (N_\phi - 1) + \left(\frac{R}{a}\right)^{2N_\phi}(N_\phi + 1)\right] \tag{52}$$

The value of R/a to be used in Eq. (52) is given by Eq. (48). Eq. (52) gives the value of S at any radius r in the medium.

When the values of S and σ_r at any radius are known, one can calculate the circumferential strain ε_θ at radius r using the following relation:

$$\varepsilon_\theta = \frac{1}{E}\left[S + (1 - \mu)\,\sigma_r\right] \qquad (53)$$

The tangential plastic strain component at any radius r in the plastic zone can be estimated from the relationship:

$$\varepsilon_p = \frac{1}{E}(S - P) \qquad (54)$$

where S and P are given by Eqs. (9) and (39) respectively.

Eqs. (52) and (39) can be combined into a single equation to yield the S - P relationship applicable in the plastic region:

$$\frac{S}{P} = \frac{N_\phi - 1}{2N_\phi} + \frac{N_\phi + 1}{2N_\phi}\left(\frac{R}{r}\right)^{2N_\phi} \qquad (55)$$

The corresponding relationship applicable to the elastic region can be readily derived from Eqs. (17) and (37):

$$\frac{S}{P} = \left(\frac{R}{r}\right)^{N_\phi + 1} \qquad (56)$$

Eq. (47) yields

$$P_o + T = \frac{N_\phi + 1}{2\,(N_\phi - 1)}\; P_R \qquad (57)$$

Let us now define $$Q = p_o + T \qquad (58)$$

and $$K = \frac{(N_\phi + 1)}{2\,(N_\phi - 1)} \qquad (59)$$

Thus Eq. (57) becomes

$$Q = K \cdot P_R$$

According to Eq. (37)

$$P = P_R \cdot \left(\frac{r}{R}\right)^{N_\phi - 1}$$

$$\therefore \qquad Q = K \cdot P_R = P \cdot K \left(\frac{R}{r}\right)^{N_\phi - 1}$$

or $$\frac{Q}{P} = K \left(\frac{R}{r}\right)^{N_\phi - 1} \qquad (60)$$

It is to be noted that the quantity Q is independent of the radial stress whether the stress situation is plastic or elastic.

Graphical Solution

Although the various quantities of interest for design can be calculated from the relationships given above, charts summarizing the relationship between P, S, and Q can be made to facilitate calculations, Newmark (1969). Examples of such charts are given in Figs. 3 through 7 for values of N_ϕ ranging from 2 to 6. These charts are based on Eqs. (55), (56) and (60).

The abscissa in the chart is S and the ordinate is P, both being plotted to a logarithmic scale. Lines sloping up to the right represent constant values of R/r and the lines sloping up to the left give constant values of Q. The heavy line for R/r = 1.00 represents the limit of elastic behavior. Below this line the behavior is plastic and above this line the behavior is elastic. If the circumferential strain and the radial stress are known at any point with radius r, then both S and P at that point are known and their intersection can be determined This intersection, if below the heavy line for R/r = 1.00, gives immediately the value of Q from which p_o may be determined. The value of R/r is also obtained, which can be used directly to determine the radius of the elasto-plastic boundary. If the point of intersection is in the elastic region, the original value of P is not valid but p_o can be determined directly from Eq. (16a) since σ_r and S are known to begin with.

Similarly if the free-field stress p_o and the radial stress at any radius r are known, the quantities Q and P can be calculated and these values can then be used to define the initial point in the chart. If the plotted point is in the plastic region, it gives immediately the values of R/r and S from which the radius of the elasto-plastic boundary and the circumferential strain ε_θ at radius r can be determined. If the plotted point is in the elastic region, the plotted value of P is not valid but the value of S is always valid. This value of S can then be used to determine the circumferential strain at the given radius r.

If we now proceed from a given radius r to some new value of r, say r = b, with R known, then R/b can be calculated and this value read as the new value of R/r. One proceeds to this new value along a line of constant Q since p_o does not change in the same material. This new point gives directly the value of S as the abscissa and the value of P as the ordinate for the new radius. If the intersection is in the elastic range of the chart, the value of P is not valid but the

value of S is always valid. If the intersection is in the plastic region, the new P is valid and σ_r can be determined directly because σ_u is known. With σ_r determined, ϵ_θ can be determined directly from S. If the new P is not valid, or the situation is elastic, σ_r can be determined from Eq. (16a) since both S and p_o are known for the new radius. Eq. (53) can then be used to determine the circumferential strain ϵ_θ.

Regardless of whether the starting point is in the elastic region or in the plastic region, a new point at another radius can be determined by going along a line of constant Q, taking into account of the fact that the change in radius is one which corresponds to a constant value of R, which can be determined from the first point located. It is thus apparent that one can start at any point and go to any other point in the same medium directly but one must start over again in each new medium, with some valid starting point being known.

The preparation of a chart for a given value of N_ϕ involves a large number of repetitive calculations. Thus, a small computer program was written and used to generate the data necessary for construction of the charts shown in Figs. 3 through 7 which were prepared for values of N_ϕ = 2, 3, 4, 5, and 6. If the value of N_ϕ in a given design problem is not an integer, the calculations may be done by interpolation using the appropriate charts. The range of input parameters covered by the charts shown in Figs. 2 through 6 is large enough to include nearly all problems involving deep protective structures in rock.

Example 1

Data:

Unlined Tunnel:

$$a = 8'$$

$$p_o = 16{,}400 \text{ psi}$$

$$E = 6x10^6 \text{ psi}$$

$$\mu = 1/3$$

$$\sigma_u = 2000 \text{ psi}$$

$$N_\phi = 4 \qquad \text{or} \qquad \phi = 37^\circ$$

$$p_i = 0$$

Required: $\epsilon_\theta \Big|_{r = a} = ?$

Solution:

$$T = \frac{2000}{4-1} = 667 \text{ psi} \tag{41}$$

At r = a

$$\frac{S}{667} = \frac{3}{8}\left[3 + 5\left(\frac{R}{a}\right)^8\right] \tag{52}$$

$$\frac{R}{a} = \left(\frac{2}{5} \times \frac{17067}{667}\right)^{1/3} \tag{48}$$

$$= 2.17$$

and

$$S = 667 \times \frac{3}{8}\left[3 + 5 \times 2.17^8\right]$$

$$= 625{,}000 \text{ psi}$$

$$\epsilon_\theta = \frac{1}{E}\left(S + (1-\mu)\ p_i\right) = \frac{625{,}000}{6 \times 10^6}$$

$$= \underline{1.04 \times 10^{-1} \quad \text{or} \quad 10.4\ \%}$$

<u>Graphical Solution (Use Chart for $N_\phi = 4$)</u>

$$p_o + T = Q = 17{,}067 \text{ psi}$$

$$P\Big|_{r=a} = \sigma_u + 3 \cdot p_i = 2000 \text{ psi}$$

For $Q = 17{,}067$ psi and $P = 2000$ psi, read from the chart for $N_\phi = 4$,

$$\frac{R}{r} = 2.17 \qquad \text{and} \qquad S = 625{,}000$$

$$\epsilon_\theta\Big|_{r=a} = \frac{1}{E}\left(S + (1-\mu)\ p_i\right)$$

$$= 10.4\ \%.$$

Case 2. <u>Input Parameters are the Same as for Case 1 except that the value of σ_u in the plastic zone varies linearly from zero at the tunnel wall (r=a) to the full value at the elasto-plastic boundary (r=R)</u>

The assumption of a linear variation of σ_u in the plastic region introduces the following changes in the relationships derived for the plastic region.

The Coulomb-Navier yield criterion for the plastic region becomes modified as

$$\sigma_\theta = \sigma_r \cdot N_\phi + \sigma_u \left(\frac{r - a}{R - a}\right) \tag{33a}$$

where R = radius of elasto-plastic boundary

The stress difference, P, at radius r in the plastic region is now given by

$$P = \sigma_\theta - \sigma_r = \sigma_r (N_\phi - 1) + \left(\frac{\sigma_u}{R - a}\right) (r - a) \tag{34a}$$

Differentiating Eq. (34a) with respect to r,

$$\frac{\partial P}{\partial r} = (N_\phi - 1) \frac{\partial \sigma_r}{\partial r} + \left(\frac{\sigma_u}{R - a}\right) \tag{35a}$$

Substituting Eq. (35a) into Eq. (1)

$$\frac{\partial P}{\partial r} - (N_\phi - 1) \frac{P}{r} = \frac{\sigma_u}{R - a} \tag{36a}$$

The solution of Eq. (36a) yields

$$P = P_a \left(\frac{r}{a}\right)^{N_\phi - 1} + \frac{\sigma_u \cdot a}{(R - a)(N_\phi - 2)} \left[\left(\frac{r}{a}\right)^{N_\phi - 1} - \frac{r}{a}\right] \tag{37a}$$

where $P_a = P\Big|_{r=a} = p_i (N_\phi - 1)$

Therefore

$$P = p_i (N_\phi - 1) \left(\frac{r}{a}\right)^{N_\phi - 1} + \frac{\sigma_u \cdot a}{(R - a)(N_\phi - 2)} \left[\left(\frac{r}{a}\right)^{N_\phi - 1} - \frac{r}{a}\right] \tag{39a}$$

At any radius r within the plastic zone, $(a \leq r \leq R)$

$$\sigma_r = \left(P - \sigma_u \cdot \frac{r - a}{R - a}\right) \Big/ (N_\phi - 1) \tag{40a}$$

$$\sigma_\theta = \sigma_r \cdot N_\phi + \sigma_u \left(\frac{r - a}{R - a}\right) \tag{33a}$$

Substitution of Eq. (39a) into Eq. (32) yields

$$r \frac{\partial S}{\partial r} + S\,(N_\phi + 1) = A\,r^{N_\phi - 1} + B\,r \tag{42a}$$

$$\left.\begin{aligned} \text{where} \quad A &= \frac{(N_\phi - 1)^2}{a^{N_\phi - 1}} \left[p_i + \frac{T \cdot a}{(R - a)(N_\phi - 2)}\right] \\ \text{and} \quad B &= \frac{\sigma_u \cdot (N_\phi - 1)}{(R - a)(2 - N_\phi)} \end{aligned}\right\} \tag{43a}$$

Solution of Eq. (42a) is of the form

$$S\,r^{N_\phi + 1} = \left(\frac{A}{2N_\phi}\right) r^{2N_\phi} + \frac{B \cdot r^{N_\phi + 2}}{N_\phi + 2} + C \tag{44a}$$

where C is the constant of integration.

At $r = R$, $\quad S = S_R$

Therefore $\quad S_R \cdot R^{N_\phi + 1} = \left(\frac{A}{2N_\phi}\right) R^{2N_\phi} + \frac{B \cdot R^{N_\phi + 2}}{N_\phi + 2} + C$

Eliminating C from the above equation, Eq. (44a) can be rewritten as

$$S \cdot r^{N_\phi + 1} = \frac{A}{2N_\phi}\left(r^{2N_\phi} - R^{2N_\phi}\right) + \frac{B}{N_\phi + 2}\left(r^{N_\phi + 2} - R^{N_\phi + 2}\right) + S_R \cdot R^{N_\phi + 1} \tag{51a}$$

At the elasto-plastic boundary (r=R)

$$P_R = \frac{2}{N_\phi + 1} (p_o + T)(N_\phi - 1) \tag{47}$$

According to Eq. (39a)

$$P_R = p_i (N_\phi - 1)\left(\frac{R}{a}\right)^{N_\phi - 1} + \frac{\sigma_u \cdot a}{(R - a)(N_\phi - 2)}\left[\left(\frac{R}{a}\right)^{N_\phi - 1} - \frac{R}{a}\right] \tag{45a}$$

Equating Eqs. (47) and (45a)

$$\left(\frac{R}{a}\right)^{N_\phi - 1}\left[p_i + \frac{T}{\left(\frac{R}{a} - 1\right)(N_\phi - 2)}\right] - \frac{R}{a}\left[\frac{T}{\left(\frac{R}{a} - 1\right)(N_\phi - 2)}\right] = \frac{2(p_o + T)}{(N_\phi + 1)} \tag{48a}$$

There is no closed solution of Eq. (48a) for R/a. Eq. (48a) has to be solved for R/a only by successive approximation.

$$\text{At} \quad r = R, \quad S_R = P_R = \frac{2 (p_o + T)(N_\phi - 1)}{(N_\phi + 1)} \tag{47}$$

Substituting this value of S_R into Eq. (51a) and simplifying

$$S\, r^{N_\phi + 1} = \frac{A}{2N_\phi}\left(r^{2N_\phi} - R^{2N_\phi}\right) + \frac{B}{N_\phi + 2}\left(r^{N_\phi + 2} - R^{N_\phi + 2}\right)$$

$$+ \frac{2 (p_o + T)(N_\phi - 1)}{N_\phi + 1} \cdot R^{N_\phi + 1} \tag{52a}$$

where A, B, and R are given by Eqs. (43a) and (48a). Eq. (52a) gives the value of S at any radius r in the medium. The circumferential strain ε_θ can then be calculated using Eq. (53).

In the elastic region ($r \geq R$) the stresses and strains are given by Eqs. (19), (20), (5), (6) and (46).

Example 2

Data:

Unlined Tunnel:

$$a = 8'$$
$$p_o = 2000 \text{ psi}$$
$$E = 6\text{x}10^6 \text{ psi}$$
$$\mu = 1/3$$
$$\sigma_u = 1200 \text{ psi}$$
$$N_\phi = 4$$
$$p_i = 0$$

Required: $\epsilon_\theta \Big|_{r=a} = ?$

Solution:

$$T = \frac{1200}{4-1} = 400 \text{ psi} \qquad (41)$$

$$\left(\frac{R}{a}\right)^3 \left(\frac{400}{\left(\frac{R}{a} - 1\right) 2}\right) - \frac{R}{a}\left(\frac{400}{\left(\frac{R}{a} - 1\right) 2}\right) = \frac{2 \text{ x } 2400}{5} \qquad (48a)$$

Solving $\frac{R}{a} = 1.75$ or $R = 1.75 \text{ x } 8 = 14'$

Note: If σ_u is constant over the plastic region

$$\frac{R}{a} = \left(\frac{2 \text{ x } 2400}{5 \text{ x } 400}\right)^{1/3} \qquad (48)$$

$$\therefore \quad \frac{R}{a} = 1.34$$

$$\left.\begin{aligned} A &= \frac{9}{a^3}\left(\frac{400}{0.75 \times 2}\right) = \frac{2400}{a^3} \\ B &= -\frac{1200 \cdot 3}{a \cdot 0.75 \cdot 2} = -\frac{2400}{a} \end{aligned}\right\} \tag{43a}$$

At $r = a$

$$S\, a^5 = \frac{2400}{a^3 \cdot 8}\left[a^8 - R^8\right] - \frac{2400}{6 \times a}\left[a^6 - R^6\right] + \frac{2 \times 2400 \times 3}{5} R^5 \tag{52a}$$

Substituting $\frac{R}{a} = 1.75$

$$S\Big|_{r=a} = 300\left[1 - 1.75^8\right] - 400\left[1 - 1.75^6\right] + 28800 \times 1.75^5$$

$$= 32200 \text{ psi}$$

$$\varepsilon_\theta\Big|_{r=a} = \frac{1}{E}\left[S + (1 - \mu)\, p_i\right] = \underline{5.4 \times 10^{-3}}$$

Note: If σ_u is constant over the plastic zone

$$S\Big|_{r=a} = 400 \times \frac{3}{8}\left[3 + 5 \times 1.34^8\right] = 8300 \text{ psi} \tag{52}$$

and $$\varepsilon_\theta\Big|_{r=a} = \frac{1}{E}\left(S + (1 - \mu)\, p_i\right) = \underline{1.4 \times 10^{-3}}$$

Case 3. Input parameters are the same as for Case 1 except that cohesion is assumed to be lost in the entire plastic region, i.e., $\sigma_u = 0$

The assumption of a complete loss of cohesion in the plastic region results in the following modified equations:

Coulomb-Navier failure criterion.

$$\sigma_\theta = \sigma_r \cdot N_\phi \tag{33b}$$

Stress difference $P = \sigma_r (N_\phi - 1)$ (34b)

$$P = P_a \cdot \left(\frac{r}{a}\right)^{N_\phi - 1} \tag{37b}$$

where $P_a = P\Big|_{r=a} = p_i (N_\phi - 1)$

At any radius r within the plastic region

$$\sigma_r = P/(N_\phi - 1) \tag{40b}$$

$$\sigma_\theta = \sigma_r \cdot N_\phi \tag{33b}$$

At $r = R$, $P_R = \frac{2}{N_\phi + 1} p_o (N_\phi - 1)$ (47b)

$$= p_i (N_\phi - 1) \left(\frac{R}{a}\right)^{N_\phi - 1} \tag{45b}$$

Thus $\frac{R}{a} = \left(\frac{2}{N_\phi + 1} \frac{p_o}{p_i}\right)^{1/N_\phi - 1}$ (48b)

Eq. (52) is now simplified as

$$S \left(\frac{r}{a}\right)^{N_\phi+1} = p_i \frac{(N_\phi - 1)}{2N_\phi} \left[\left(\frac{r}{a}\right)^{2N_\phi} (N_\phi - 1) + \left(\frac{R}{a}\right)^{2N_\phi} (N_\phi + 1)\right] \qquad (52b)$$

Eq. (52b) can now be used to determine the tangential strain ϵ_θ at radius r. In the elastic region, $r \geq R$, the stresses and strains are determined using Eqs. (19), (20), (5) and (6) noting that $\sigma_R = 2\, p_o/(N_\phi + 1)$.

Case 4. Input parameters are the same as Case 1 except that the quantities E and μ have different values in the elastic and plastic regions.

Let

E, E' = Young's modulus in the elastic and plastic regions

μ, μ' = Poisson's ratio in the elastic and plastic regions

k = E/E'

The tangential strain at the elasto-plastic boundary (r=R) must have the same value in both the elastic and plastic regions. Since the values of the Young's modulus are different in the two regions, there will be a discontinuity in the tangential stress at r = R.

Let σ_R be the radial stress at the elasto-plastic boundary (r=R)

The circumferential strain at r = R due to stresses in the elastic zone is

$$\epsilon_{\theta R} = \frac{1}{E'} \left[\sigma_R \cdot N_\phi + \sigma_u - \mu' \sigma_R\right]$$

The corresponding value at $r = R$ due to stresses in the elastic region is

$$\epsilon_{\theta R} = \frac{1}{E}\left[(2\,p_o - \sigma_R) - \mu\sigma_R\right]$$

Equating these two expressions for $\epsilon_{\theta R}$

$$\sigma_R = \frac{2\,p_o - k\,\sigma_u}{k\,(N_\phi - \mu') + (1 + \mu)} \qquad (46c)$$

According to Eq. (40) σ_R is also given by

$$\sigma_R = (p_i + T)\left(\frac{R}{a}\right)^{N_\phi - 1} - T \qquad (40c)$$

Equating Eqs. (46c) and (40c) and solving for R/a:

$$\frac{R}{a} = \left[\left\{\frac{2\,p_o - k\,\sigma_u}{k\,(N_\phi - \mu') + (1 + \mu)} + T\right\} \Big/ (p_i + T)\right]^{1/N_\phi - 1} \qquad (48c)$$

Eq. (48c) thus gives the radius of the elasto-plastic boundary.

<u>For $r \geq R$ (elastic zone)</u>

$$\sigma_r = p_o - (p_o - \sigma_R)\left(\frac{R}{r}\right)^2 \qquad (19)$$

$$\sigma_\theta = p_o + (p_o - \sigma_R)\left(\frac{R}{r}\right)^2 \qquad (20)$$

$$\epsilon_r = \frac{1}{E}(\sigma_r - \mu\sigma_\theta) \qquad (6)$$

$$\epsilon_\theta = \frac{1}{E}(\sigma_\theta - \mu\sigma_r) \qquad (5)$$

For $a \leq r \leq R$ (plastic zone)

$$\sigma_r = (p_i + T) \left(\frac{r}{a}\right)^{N_\phi - 1} - T \qquad (40)$$

$$\sigma_\theta = N_\phi \cdot \sigma_r + \sigma_u \qquad (33)$$

The circumferential strain ϵ_θ in the plastic region can be obtained from S which is given by Eq. (52):

$$\frac{S}{p_i + T} \left(\frac{r}{a}\right)^{N_\phi + 1} = \frac{N_\phi - 1}{2N_\phi} \left[(N_\phi - 1) \left(\frac{r}{a}\right)^{2N_\phi} + (N_\phi + 1) \left(\frac{R}{a}\right)^{2N_\phi} \right] \qquad (52)$$

$$\epsilon_\theta = \frac{1}{E'} \left[S + (1 - \mu') \sigma_r \right] \qquad (53c)$$

Graphical Solution

The solution to Case 4 can also be obtained using the chart. In this case, since the elastic and plastic zones are assumed to have different material constants, they will be treated as two different media arranged in a concentric manner. For each medium (in this case for each zone) the appropriate chart will have to be used.

Using the given data, the initial starting point is located in the chart of one medium. Then the radial stress and the tangential strain at the boundary between the two media are determined which are then used to locate the initial starting point in the second medium. The same procedure can be repeated if there are more than two different media to deal with.

If the circumferential strain and the radial stress are known at any radius r in the medium, the procedure described above can be used directly to

estimate the free field pressure p_o or the modulus of the medium required to limit the circumferential strain at the tunnel wall to a tolerable value under a given value of p_o. But very often the data may be insufficient to locate the initial point in the chart for a graphical analysis, as for example, when it is required to estimate the circumferential strain at r = a in Case 4 for a given value of p_o. Under these conditions, it becomes necessary to use a successive approximation procedure. For the example cited, an initial value of circumferential strain at r = a is assumed and the graphical analysis is carried outward from the innermost radius a and the value of p_o is estimated. If this value agrees with the value given in the problem, the estimate of the circumferential strain at r = a is correct. If not, a revised value of the circumferential strain is assumed and the procedure is repeated until the assumed circumferential strain is consistent with the given data.

Example 3

Data:

Unlined Tunnel:

$$
\begin{aligned}
a &= 8' \\
p_o &= 2000 \text{ psi} \\
E &= 6\text{x}10^6 \text{ psi} \\
E' &= 3\text{x}10^6 \text{ psi} \\
\mu &= 1/3 \\
\mu' &= 1/3 \\
\sigma_u &= 1200 \text{ psi} \\
N_\phi &= 4 \\
p_i &= 0
\end{aligned}
$$

Required: $\epsilon_\theta \Big|_{r=a} = ?$

Solution:

$$T = \frac{1200}{4-1} = 400 \text{ psi} \qquad (41)$$

$$k = E/E' = 2$$

$$\frac{R}{a} = \left[\left(\frac{4000 - 2400}{2 \times 3\ 2/3 + 1\ 1/3} + 400\right) \Big/ 400\right]^{1/3} \qquad (48c)$$

$$= 1.136$$

(Compare with the value 1.34 obtained for Example 2 with E = E' and σ_u constant).

At r = a

$$\frac{S}{400} = \frac{3}{8}\left[3 + 5\left(\frac{R}{a}\right)^8\right] \qquad (52)$$

$$= \frac{3}{8}\left[3 + 5\,(1.136)^8\right]$$

$$S = 2540 \text{ psi.}$$

$$\epsilon_\theta = \frac{1}{3\times10^6}\,[2540 + 2/3 \times 0] = 8.5 \times 10^{-4}$$

(Compare with the value 14×10^{-4} obtained for Example 2 with E = E' and σ_u = constant).

Graphical Solution

The data given in the example problem does not enable the starting point to be located in the chart for $N_\phi = 4$ for the plastic zone. This means that the problem has to be solved by successive approximation.

Let us assume that $\varepsilon_\theta \Big|_{r=a} = 8.5 \times 10^{-4}$

At r = a,

$$S = E' \varepsilon_\theta - (1 - \mu') p_i = 3x10^6 \times 8.5 \times 10^{-4} - 0$$

$$= 2550 \text{ psi}$$

$$P = \sigma_u + (N_\phi - 1) p_i = 1200 \text{ psi.}$$

For S = 2550 psi and P = 1200 psi, the chart (N_ϕ=4) gives

$$\frac{R}{r} = \frac{R}{a} = 1.135 \qquad \text{or} \qquad R = 1.135\ a$$

and $Q = 1480$ psi

At r = R, $\frac{R}{r} = 1.0$

For R/r = 1.0 and Q = 1480 psi, the chart gives

$$P_R = 1755; \qquad S_R = 1755 \text{ psi}$$

$$P_R = \sigma_u + 3\,\sigma_R = 1200 + 3\,\sigma_R$$

$$\therefore \quad \sigma_R = 185 \text{ psi}$$

$$S_R = E' \epsilon_{\theta R} - (1 - \mu') \sigma_R$$

or $$\epsilon_{\theta R} = \frac{1}{E'} \left[S_R + (1 - \mu') \sigma_R \right]$$

$$= \frac{1}{3x10^6} \left[1755 + \frac{2}{3} x 185 \right]$$

$$= 626 \times 10^{-6}$$

Now in the elastic region

$$S_R = E \epsilon_{\theta R} - (1 - \mu) \sigma_R$$

$$= 6x10^6 \times 626x10^{-6} - \frac{2}{3} x 185$$

$$= (3756 - 123) 10 = 3633 \text{ psi}$$

$$2p_o = S_R + \sigma_R = 4003 \text{ psi}$$

$$p_o = 2001.5 \simeq 2000 \text{ psi (given)}$$

This means that the initial assumed value of $\epsilon_\theta \Big|_{r=a}$ is correct.

If the value of p_o obtained for an assumed value of $\epsilon_\theta \Big|_{r=a}$ does not agree with the given value of p_o, the procedure has to be repeated with a second trial value for $\epsilon_\theta \Big|_{r=a}$. This repetitive procedure is continued until all the input data are satisfied.

Case 5. <u>In this case the tunnel is assumed to be initially surrounded with a circular zone of loosened material having elastic constants which are different from those of the insitu material</u>.

The case could represent a tunnel with a "destressed zone" from static loading which is subsequently loaded with another free-field stress such as a dynamic loading over a large area.

Let a = radius of tunnel opening

R = radius of elasto-plastic boundary

R_1 = radius of loosened zone

R_2 = radius of elasto-plastic boundary in the loosened medium, if any, when $R > R_1$

p_o = uniform free-field pressure

p_i = internal pressure at $r = a$

E = Young's modulus of intact medium

E' = Young's modulus of loosened medium

μ = Poisson's ratio of intact medium.

μ' = Poisson's ratio of loosened medium.

σ_u = unconfined compressive strength of the medium (assumed to be the same in both the intact and loosened zones)

$N_\phi = \dfrac{1 + \sin\phi}{1 - \sin\phi}$

ϕ = angle of shearing resistance of the medium (assumed to be the same in both the intact and loosened zones)

In the analysis of the present problem, two distinct cases have to be considered:

a) $R \leq R_1$ b) $R > R_1$

Case (a) $R \leq R_1$

The elastic zone ($r \geq R$) in this case consists of two rings of material with two different values of Young's modulus. Therefore the stresses and strains in the elastic zone can be computed using the elastic solution obtained by Savin (1961).

For $R \leq r \leq R_1$

$$\sigma_r = \sigma_R + (p_o - \sigma_R) \; C_1 \tag{61}$$

$$\sigma_\theta = \sigma_R + (p_o - \sigma_R) \; C_2 \tag{62}$$

where

$$C_1 = \left[(1 + \chi_2) \; (n^2 - R_1{}^2/r^2)\right] \Big/ D \tag{63}$$

$$C_2 = \left[(1 + \chi_2) \; (n^2 + R_1{}^2/r^2)\right] \Big/ D \tag{64}$$

$$D = 2\,(\alpha - 1) - n^2 \left[(\alpha - 1) - (1 + \chi_1 \; \alpha)\right] \tag{65}$$

$$\chi_1 = \frac{3 - \mu'}{1 + \mu'} \tag{66}$$

$$\chi_2 = \frac{3 - \mu}{1 + \mu} \tag{67}$$

$$n = R_1/R \tag{68}$$

$$\alpha = \frac{E\,(1 + \mu')}{E'\,(1 + \mu)} \tag{69}$$

and

$$\sigma_R = \sigma_r \Big|_{r=R}$$

For $r \geq R_1$

$$\sigma_r = p_o - (p_o - \sigma_R) \; C_3 \tag{70}$$

$$\sigma_\theta = p_o + (p_o - \sigma_R) \; C_3 \tag{71}$$

where
$$C_3 = \left[1 - \frac{(n^2 - 1)\;(1 + \chi_2)}{D}\right] \frac{R_1{}^2}{r^2} \tag{72}$$

The strains in the elastic zone can be obtained from the stresses as follows:

For $R \leq r \leq R_1$
$$\varepsilon_r = \frac{1}{E'} \; (\sigma_r - \mu' \; \sigma_\theta) \tag{73}$$

$$\varepsilon_\theta = \frac{1}{E'} \; (\sigma_\theta - \mu' \; \sigma_r) \tag{74}$$

For $r \geq R_1$
$$\varepsilon_r = \frac{1}{E} \; (\sigma_r - \mu \; \sigma_\theta) \tag{75}$$

$$\varepsilon_\theta = \frac{1}{E} \; (\sigma_\theta - \mu \; \sigma_r) \tag{76}$$

However, in Eqs. (61), (62), (70) and (71), σ_R is still an unknown and therefore has to be obtained before the stresses and strains can be determined in the elastic region. This can be done by equating the circumferential strains at $r = R$ in both the elastic and plastic regions. Thus

$$\frac{1}{E'} \; (\sigma_{\theta Re} - \mu' \; \sigma_R) = \frac{1}{E'} \; (\sigma_{\theta Rp} - \mu' \; \sigma_R) \tag{77}$$

where $\sigma_{\theta Re} = \sigma_\theta \Big|_{r=R}$ in the elastic region

$\sigma_{\theta Rp} = \sigma_\theta \Big|_{r=R}$ in the plastic region

Eq. (77) gives $\sigma_{\theta Re} = \sigma_{\theta Rp}$

By Eq. (62) $\sigma_{\theta Re} = \sigma_R + (p_o - \sigma_R) \; C_2\Big|_{r=R}$

$$= \sigma_R + (p_o - \sigma_R) \; C \tag{78}$$

where $C = (1 + \chi_2) \cdot 2 \; n^2/D$

In the plastic region

$$\sigma_{\theta Rp} = \sigma_R \cdot N_\phi + \sigma_u \tag{79}$$

Equating Eqs. (78) and (79), and simplifying

$$\sigma_R = \left(\frac{C \; p_o - \sigma_u}{N_\phi - 1 + C}\right) \tag{80}$$

Eq. (80) can now be used in conjunction with Eqs. (61) through (76) to define completely the stresses and strains in the elastic zone.

From Eq. (40) it is known that

$$\sigma_R = (p_i + T) \left(\frac{R}{a}\right)^{N_\phi - 1} - T \tag{81}$$

Eqs. (80) and (81) can now be combined to yield

$$\frac{R}{a} = \left[\left(\frac{C\ p_o - \sigma_u}{N_\phi - 1 + C} + T\right) \Big/ (p_i + T)\right]^{1/N_\phi - 1} \tag{82}$$

The value of R/a obtained from Eq. (82) should be less than R_1/a for the above analysis to be valid. The analysis of the problem when $R > R_1$ is considered under Case (b).

The stresses in the plastic zone ($a \leq r \leq R$) are given by Eqs. (40) and (33). The circumferential strains in the plastic zone can be determined from Eq. (52) and

$$S = E'\ \varepsilon_\theta - (1 - \mu')\ \sigma_r \tag{83}$$

Case (b) $R > R_1$

When $R > R_1$, the loosened zone may exist in any one of the following three states, depending on the magnitude of the radial stress σ_{R1} at $r = R_1$:

(i) Completely plastic

(ii) Partly plastic and partly elastic

(iii) Completely elastic

Therefore it is necessary to first determine the magnitude of σ_{R1}. The procedure for determining σ_{R1} is given below.

Based on Eq. (48) it may be written that

$$\left(\frac{R}{R_1}\right)^{N_\phi - 1} = \frac{2}{N_\phi + 1} \cdot \frac{p_o + T}{\sigma_{R1} + T} \tag{84}$$

Similarly from Eq. (52) it can be derived that

$$\frac{S_{R1}}{\sigma_{R1} + T} = \frac{N_\phi - 1}{2N_\phi} \left[(N_\phi - 1) + (N_\phi + 1)\left(\frac{R}{R_1}\right)^{2N_\phi} \right] \tag{85}$$

where $S_{R1} = E\, \epsilon_{\theta R1} - (1 - \mu)\, \sigma_{R1}$ (86)

and $\epsilon_{\theta R1} = \epsilon_\theta \Big|_{r=R1}$.

Eqs. (84), (85) and (86) can be combined to give

$$\epsilon_{\theta R1} = \frac{1}{E} \left[(\sigma_{R1}+T) \cdot \frac{N_\phi-1}{2N_\phi} \cdot \left\{ (N_\phi-1) + (N_\phi+1) \left(\frac{2}{N_\phi+1} \cdot \frac{p_o+T}{\sigma_{R1}+T} \right)^{\frac{2N_\phi}{N_\phi-1}} \right\} + (1-\mu)\, \sigma_{R1} \right]$$

$$= \frac{K}{E} \tag{87}$$

As a general case, let it be assumed that the loosened zone becomes partly plastic and partly elastic. If R_2 is the radius of the elasto-plastic boundary in the loosened zone the following relationships must be satisfied.

$$\sigma_{R2} = \sigma_r \Big|_{r=R_2} = (p_i + T) \left(\frac{R_2}{a}\right)^{N_\phi-1} - T \tag{40d}$$

$$\sigma_{\theta R1e} = \sigma_\theta \Big|_{r=R_1} \quad \text{in the elastic loosened zone}$$

$$= \sigma_{R1} \frac{R_1^2 + R_2^2}{R_1^2 - R_2^2} - \sigma_{R2} \frac{2R^2}{R_1^2 - R_2^2} \tag{88}$$

$$\epsilon_{\theta R1} = \frac{1}{E'} (\sigma_{\theta R1e} - \mu' \sigma_{R1}) \tag{5d}$$

Equating Eqs. (87) and (5d)

$$\epsilon_{\theta R1e} - \mu' \sigma_{R1} = \frac{E'}{E} K \tag{89}$$

It can also be shown that

$$(N_\phi + 1) \sigma_{R2} + \sigma_u = 2 \frac{\sigma_{R1} \cdot R_1^2 - \sigma_{R2} \cdot R_2^2}{R_1^2 - R_2^2} \tag{90}$$

Eqs. (89), (90) and (40d) can now be solved to yield the values of the two unknowns σ_{R1} and R_2.

If the loosened zone remains entirely elastic or entirely plastic, Eqs. (40d), (88) and (90) are not valid and cannot be used to determine σ_{R1}.

When the loosened zone is entirely elastic

$$\sigma_{\theta R1e} = \frac{\sigma_{R1} (R_1^2 + a^2) - p_i \cdot 2 a^2}{R_1^2 - a^2} \tag{91}$$

This value of $\sigma_{\theta R1e}$ can now be substituted into Eq. (89) to get the value of σ_{R1}. Substitution of σ_{R1} in Eq. (84) gives the value of R.

When the loosened zone is entirely plastic, the values of R, σ_R and σ_{R1} can be obtained directly from the following equations:

$$\left.\begin{aligned} \left(\frac{R}{a}\right)^{N_\phi - 1} &= \frac{2}{N_\phi + 1} \left(\frac{p_o + T}{p_i + T}\right) \\ \sigma_R &= (2\,p_o - \sigma_u) \Big/ (N_\phi + 1) \\ \sigma_{R1} &= (\sigma_R + T) \left(\frac{R_1}{R}\right)^{N_\phi - 1} - T \end{aligned}\right\} \tag{92}$$

The value of R obtained from the preceeding equations should be greater than R_1. If not, the procedure given under case (a) has to be used.

Stresses and Strains

$r \geq R$

$$\sigma_r = p_o - (p_o - \sigma_R) \left(\frac{R}{r}\right)^2 \tag{93}$$

$$\sigma_\theta = p_o + (p_o - \sigma_R) \left(\frac{R}{r}\right)^2 \tag{94}$$

$$\varepsilon_r = \frac{1}{E} (\sigma_r - \mu\, \sigma_\theta) \tag{95}$$

$$\varepsilon_\theta = \frac{1}{E} (\sigma_\theta - \mu\, \sigma_r) \tag{96}$$

where

$$\sigma_r = \sigma_r \Big|_{r=R} = \frac{(2\,p_o - \sigma_u)}{N_\phi + 1} \tag{92}$$

$R_1 \leq r \leq R$

$$\sigma_r = \frac{2\,(p_o + T)}{N_\phi + 1} \left(\frac{r}{R}\right)^{N_\phi - 1} - T \qquad (97)$$

$$\sigma_\theta = \sigma_r \cdot N_\phi + \sigma_u \qquad (98)$$

$$\frac{S}{p_o + T} \left(\frac{r}{R}\right)^{N_\phi + 1} = \frac{N_\phi - 1}{N_\phi\,(N_\phi + 1)} \left[(N_\phi - 1) \left(\frac{r}{R}\right)^{2N_\phi} + (N_\phi + 1) \right] \qquad (99)$$

where $S = E\,\varepsilon_\theta - (1 - \mu)\,\sigma_r$

$a \leq r \leq R_1$

Fully plastic

$$\sigma_r = (p_i + T) \left(\frac{r}{a}\right)^{N_\phi - 1} - T \qquad (100)$$

$$\sigma_\theta = \sigma_r \cdot N_\phi + \sigma_u \qquad (98)$$

$$S\,r^{N_\phi + 1} = \frac{(N_\phi - 1)^2\,(p_i + T)}{2\,N_\phi \cdot a^{N_\phi - 1}} \left[r^{2N_\phi} - R^{2N_\phi} \right] + S'_{R1}\;R_1^{N_\phi + 1} \qquad (101)$$

where $S = E'\,\varepsilon_\theta - (1 - \mu')\,\sigma_r$

$S'_{R1} = E'\,\varepsilon_{\theta R1} - (1 - \mu')\,\sigma_{R1}$

and $\epsilon_{\theta R1} = \epsilon_\theta \Big|_{r=R1}$ as obtained from Eq. (99).

Partly plastic and partly elastic

$R_2 \leq r \leq R_1$

$$\sigma_r = \frac{\sigma_{R1} \cdot R_1^2 - \sigma_{R2} \cdot R_2^2}{R_1^2 - R_2^2} - \frac{(\sigma_{R1} - \sigma_{R2}) R_1^2 R_2^2}{(R_1^2 - R_2^2) r^2} \tag{102}$$

$$\sigma_\theta = \frac{\sigma_{R1} \cdot R_1^2 - \sigma_{R2} \cdot R_2^2}{R_1^2 - R_2^2} + \frac{(\sigma_{R1} - \sigma_{R2}) R_1^2 R_2^2}{(R_1^2 - R_2^2) r^2} \tag{103}$$

$$\epsilon_r = \frac{1}{E'} (\sigma_r - \mu' \sigma_\theta) \tag{104}$$

$$\epsilon_\theta = \frac{1}{E'} (\sigma_\theta - \mu' \sigma_r) \tag{105}$$

$a \leq r \leq R_2$

$$\sigma_r = (p_i + T) \left(\frac{r}{a}\right)^{N_\phi - 1} - T \tag{100}$$

$$\sigma_\theta = \sigma_r \cdot N_\phi + \sigma_u \tag{98}$$

$$\frac{S}{p_i + T} \left(\frac{r}{a}\right)^{N_\phi + 1} = \frac{N_\phi - 1}{2N_\phi} \left[(N_\phi - 1) \left(\frac{r}{a}\right)^{2N_\phi} + (N_\phi + 1) \left(\frac{R_2}{a}\right)^{2N_\phi} \right] \tag{106}$$

where $S = E' \epsilon_\theta - (1 - \mu') \sigma_r$

Fully elastic

$$\sigma_r = A - B/r^2 \tag{107}$$

$$\sigma_\theta = A + B/r^2 \tag{108}$$

where

$$A = \frac{\sigma_{R1} \cdot R_1^{\;2} - p_i \cdot a^2}{R_1^{\;2} - a^2} \tag{109}$$

$$B = \frac{(\sigma_{R1} - p_i)\, R_1^{\;2}\, a^2}{(R_1^{\;2} - a^2)} \tag{110}$$

$$\varepsilon_r = \frac{1}{E'} \left(\sigma_r - \mu' \sigma_\theta\right) \tag{104}$$

$$\varepsilon_\theta = \frac{1}{E'} \left(\sigma_\theta - \mu' \sigma_r\right) \tag{105}$$

The preceeding paragraphs thus give a complete and closed analytical solution for Case 5. This problem can, however, be solved more readily by the graphical procedure using appropriate charts. The details of the procedure are the same as explained at the end of Case 4.

Example 4

The input parameters for this example are the same as for Example 1 except that the tunnel opening is surrounded by a loosened zone 8 feet thick having the following properties:

$$E' = 2 \times 10^6 \text{ psi}$$

$$\mu' = 1/3$$

$$\sigma_u = 2000 \text{ psi}$$

$$N_\phi = 4$$

Required: $\left.\varepsilon_\theta\right|_{r=a} = ?$

Solution:

This problem can be solved analytically using the relationships derived for Case 5. But the procedure is long and time-consuming. Therefore the easier graphical procedure is used in this example.

As has been explained earlier, the solution to this problem can only be obtained by successive trials, the number of trials being dependent on how close the first trial value is to the actual value. In the present case let us assume

$$\left.\varepsilon_\theta\right|_{r=a} = 0.05$$

Then at $r = a$

$$\sigma_u = 2000 \text{ psi}$$

$$\varepsilon_\theta = 0.05$$

$$\sigma_r = 0$$

$$E = E' = 2 \times 10^6 \text{ psi}$$

$$\mu' = \mu = 1/3$$

$$\therefore \quad S = 2 \times 10^6 \times 0.05 - 0 = 10^5 \text{ psi} \qquad (9)$$

$$P = 2000 + 0 = 2000 \text{ psi} \qquad (34)$$

For these values of S and P, read from Chart ($N_\phi = 4$)

$$Q = 8600 \text{ psi}$$

$$\frac{R}{a} = 1.725$$

At $r = R_1 = 2a$ (= 16') $\qquad \frac{R}{R_1} = 0.863$

For $Q = 8600$ psi and $\frac{R}{r} = \frac{R}{R_1} = 0.863$, read from the chart

$$S\Big|_{r=R_1} \text{ (loosened medium)} = 7700 \text{ psi}$$

At $\frac{R}{R_1} < 1$, the material is elastic at $r = R_1$

$$\sigma_r\Big|_{r=R_1} = \frac{2\,p_o - S}{2} = \frac{2\,(Q - T) - S}{2} = (Q - T) - \frac{S}{2} \qquad (19)$$

$$= (8600 - 667) - \frac{7700}{2} = 4083 \text{ psi.}$$

At $r = R_1$ <u>in the loosened zone</u>,

$$\epsilon_\theta = \frac{1}{E'}\left[S + (1 - \mu')\,\sigma_r\right]$$

$$= \frac{1}{2\text{x}10^6}\left[7700 + \frac{2}{3} \text{ x } 4083\right]$$

$$= 5211 \text{ x } 10^{-6}$$

At $r = R_1$ <u>in the unloosened zone</u>

$$S = E\,\varepsilon_\theta - (1-\mu)\,\sigma_r \qquad (9)$$

$$= 6\text{x}10^6 \text{ x } 5211 \text{ x } 10^{-6} - \frac{2}{3} \text{ x } 4083$$

$$= 28544 \text{ psi}$$

$$P = \sigma_u + (N_\phi - 1)\,\sigma_r = 2000 + 3 \text{ x } 4083 \qquad (34)$$

$$= 14250 \text{ psi}$$

For these values of S and P, the chart for $N_\phi = 4$ gives

$$Q = 16900 \quad \text{or} \quad p_o = Q - T = 16900 - 667 = 16233 \text{ psi}$$

This value is sufficiently close to the given value of p_o of 16400 psi. So the assumed value of $\varepsilon_\theta\big|_{r=a} = 0.05$ is very near the correct value and the calculations need not be repeated.

Case 6. Lined tunnel with a circular zone of loosened material.

A tunnel may be provided with a liner to limit the strains at the inner surface of the tunnel within allowable limits. The linings may be of concrete or steel or a combination of both. In this analysis the liner materials also will be assumed to behave elasto-plastically. The analytical solution of the stresses and strains in a tunnel system consisting of one or more sets of liners and a loosened zone of material surrounding the liners, is very complex, lengthy and cumbersome. However, the graphical solution, using the charts of the type presented earlier, offers a relatively simple means of solving the problem.

The analysis must proceed outward from an inner element or inward from an outer element of the tunnel system. For each element of the system the chart with the appropriate value of N_ϕ has to be used. At least two quantities, such as the circumferential strain or the radial stress, are required to locate the initial starting point for the analysis of any one element. Once these quantities are known, they can be used to estimate the values of the radial stress and circumferential strain at the boundary with the next element, which are in turn used to locate the starting point for the analysis of that next element. This process can be repeated until the stresses and strains in the whole tunnel system are known. The following example is worked out to illustrate the approach mentioned above.

Example 5

The dimensions and rock properties assumed in this example (Fig. 8) are the same as for Example 4 except that the tunnel opening is provided with a 12" thick liner of concrete having the following properties:

$$E_c = 4\text{x}10^6 \text{ psi}$$

$$\mu_c = 1/3$$

$$\sigma_u = 5000 \text{ psi}$$

$$N_\phi = 4$$

Required: $\varepsilon_\theta \Big|_{r=a} = ?$

Solution:

Let us consider the concrete liner first. Let us assume, as a first trial value,

$$\varepsilon_\theta \Big|_{r=7'} = 0.03$$

It is known that at r = 7'

$$\sigma_r = 0$$

$$S = 4 \times 10^6 \times 0.03 - 0 = 1.2 \times 10^5 \text{ psi} \qquad (9)$$

$$P = 5000 + 0 = 5000 \text{ psi} \qquad (34)$$

For these values of S and P, read from chart [N_ϕ(concrete) = 4]

$$Q = 16500 \text{ psi}$$

$$\frac{R}{7} = 1.57, \quad \text{thus} \quad R = 11'$$

At r = 8' $\frac{R}{r} = \frac{11}{8} = 1.375$

For $Q = 16500$ psi and $\frac{R}{r} = 1.375$, read

$$S = 64000 \text{ psi}$$

$$P = 7600 \text{ psi} = 5000 + 3\,\sigma_r \qquad (34)$$

Therefore at r = 8'

$$\therefore \sigma_r = 867 \text{ psi}$$

$$\epsilon_\theta = \frac{1}{4x10^6} [64000 + 2/3 \times 867] \qquad (53)$$

$$= 16147 \times 10^{-6}$$

Let us now consider the loosened zone.

At r = 8'

$$\epsilon_\theta = 16147 \times 10^{-6} \quad \text{and}$$

$$\sigma_r = 867 \text{ psi}$$

$$\therefore S = 2 \times 10^6 \times 16147 \times 10^{-6} - 2/3 \times 867 \qquad (9)$$

$$= 31720 \text{ psi}$$

$$P = 2000 + 3 \times 867 = 4600 \text{ psi} \qquad (34)$$

For these values of S and P, read from the $N_\phi = 4$ chart for the loosened zone,

$$Q = 9250 \text{ psi}$$

$$\frac{R}{8} = 1.34 \qquad \therefore \quad R = 10.72'$$

At r = 16' $\quad \frac{R}{r} = \frac{10.72}{16} = 0.67 < 1$

For Q = 9250 psi and $\frac{R}{r} = 0.67$, read

$$S = 5000 \text{ psi}$$

As $\frac{R}{r} = 0.67 < 1$, the value of P is not valid, but σ_r can be calculated using the relation

$$\sigma_r = p_o - \frac{S}{2} \text{ where } p_o = Q - T$$

Thus at r = 16'

$$\therefore \quad \sigma_r = (Q - T) - \frac{S}{2} = (9250 - 667) - \frac{5000}{2}$$

$$= 6083 \text{ psi}$$

$$\epsilon_\theta = \frac{1}{2x10^6}\left[5000 + \frac{2}{3} \times 6083\right] \qquad (53)$$

$$= 4528 \times 10^{-6}$$

Let us now consider the intact (non loosened) medium.

At r = 16'

$$\epsilon_\theta = 4528 \times 10^{-6}$$

$$\sigma_r = 6083 \text{ psi}$$

$$\therefore \quad S = 6 \times 10^6 \times 4528 \times 10^{-6} - \frac{2}{3} \times 6083 \qquad (9)$$

$$= 23113 \text{ psi}$$

$$P = 2000 + 3 \times 6083 \qquad (34)$$

$$= 20250 \text{ psi}$$

For these values of S and P,

$$Q = 18200 \text{ psi}$$

and $$p_o = Q - T = 18200 - 667 = 17533 \text{ psi}$$

But the value of p_o specified in the problem is 16400 psi. This means that the assumed value of $\varepsilon_\theta \Big|_{r=7'} = 0.03$ is slightly on the high side.

So let us now assume that at r = 7'

$$\varepsilon_\theta = 0.028$$

A similar analysis gives the following values:

At r = 7' <u>In concrete</u>

$S = 1.12 \times 10^5$ psi $\qquad P = 5000$ psi

At r = 8' <u>In concrete</u>

$S = 5.95 \times 10^4$ $\qquad P = 7500$ psi

$\sigma_r = 833$ psi

$\varepsilon_\theta = 15014 \times 10^{-6}$

<u>In loosened zone</u>

$S = 29472$ psi $\qquad P = 4500$ psi

At r = 16' <u>In loosened zone</u>

$S = 4.7 \times 10^3$ psi

$\sigma_r = 5883$ psi

$\varepsilon_\theta = 4311 \times 10^{-6}$

In intact zone

$S = 21944$ psi $\qquad\qquad P = 19650$ psi

$p_o = 16433$ (≈ 16400) psi

Thus the assumed value of ε_θ (at $r = 7'$) $= 0.028$ is very near the actual value.

Note: For relatively thin steel liners, the analysis is slightly simplified because the radial pressure exerted by the thin steel liner against the medium next to it is given by

$$\sigma_{ra} = \frac{h_s}{a} E_s \varepsilon_\theta \qquad \text{or} \qquad \frac{h_s}{a} \cdot \sigma_y$$

whichever is less. In the above equation

σ_{ra} = radial pressure exerted by steel against the adjacent medium

h_s = thickness of steel lining

σ_y = yield stress of steel lining

a = radius to outside of steel lining

E_s = Young's modulus of steel lining

ε_θ = circumferential strain of the steel liner

Case 7. Tunnel provided with back packing: Loosened zone present.

This case is very similar to Case 5, the only difference being that the radial pressure p_i at $r = a$ is $\neq 0$. The back packing material usually has a

very low yield value and serves to exert an equal all around pressure on the inner surface of the tunnel over a considerable range of strain. The analysis for the present case is illustrated by the following example problem and is very similar to that of Example 4.

Example 6

The input parameters are the same as those of Example 4 except that the radial pressure due to back packing is 150 psi (p_i = 150 psi).

Required: $\left.\epsilon_\theta\right|_{r=a} = ?$

Solution:

Assume at $r = a = 8'$, $\epsilon_\theta = 0.04$

Then at $r = 8'$

$$S = 2 \times 10^6 \times .04 - \frac{2}{3} \times 150 = 79900 \text{ psi} \tag{9}$$

$$P = 2000 + 3 \times 150 = 2450 \text{ psi} \tag{34}$$

For these values of S and P read from chart (N_ϕ=4)

$$Q = 9000 \text{ psi} \qquad \frac{R}{8} = 1.64'$$

$$R = 13.12'$$

At $r = 16'$

$$\frac{R}{r} = \frac{13.12}{16.0} = 0.82 < 1$$

For $Q = 9000$ psi, $\frac{R}{r} = 0.82$ $S = 7250$ psi

$$\sigma_r = (Q - T) - \frac{S}{2}$$

$$= (9000 - 667) - 3625$$

$$= 4708 \text{ psi}$$

$$\varepsilon_\theta = \frac{1}{2x10^6} \left[7250 + \frac{2}{3} x 4708\right]$$

$$= 5195 \times 10^{-6}$$

Now for the intact material at $r = 16'$

$$\sigma_r = 4708 \text{ psi}$$

$$\varepsilon_\theta = 5195 \times 10^{-6}$$

$$\therefore \quad S = 6 \times 10^6 \times 5195 \times 10^{-6} - \frac{2}{3} \times 4708 \qquad (9)$$

$$= 28040 \text{ psi}$$

$$P = 2000 + 3 \times 4708 = 16124 \text{ psi} \qquad (34)$$

For these values of S and P

$$Q = 18000 \text{ psi}$$

$p_o = Q - T = 17333$ psi (in comparison with the actual value of 16400 psi)

Therefore the assumed value of ε_θ at $r = a = 8'$ has to be revised. So let us now assume at $r = 8'$ $\quad \varepsilon_\theta = 0.038$.

A similar analysis leads to a value of $p_o = 16633$ psi. So by a slight extrapolation it can be found that for $p_o = 16400$ psi,

$$\varepsilon_\theta \Big|_{r=8'} = 0.0373 \quad \text{or } 3.7\%.$$

Chapter 3

Conclusions

General

The analysis presented in the previous chapter yields a means whereby stresses and strains can be determined around a circular tunnel in a Coulomb-Navier material which increases in volume at failure. This analysis may be used to design liners if the insitu rock properties are known. The results of the analysis are dependent on the rock properties assumed; therefore, recommendations are given in this chapter on the selection of the approriate rock mass properties to use in this analysis. The effects of a non-uniform system of free-field stresses are also discussed.

Shear Strength Properties

The values of σ_u and N_ϕ to be used in the analysis are related to the properties of the jointed rock mass surrounding the opening. N_ϕ is given by $\frac{1 + \sin \phi}{1 - \sin \phi}$ where the appropriate angle of frictional resistance should be taken as the angle of frictional resistance along the joints or discontinuities not the angle of internal friction derived from triaxial tests on intact samples of rock. The value of the unconfined compressive strength of the rock mass, σ_u, is a function of the ratio of tunnel diameter to joint spacing as shown in Fig. 9. The larger the ratio of tunnel diameter to joint spacing, D/S, the smaller the value of σ_u appropriate for design. If D/S is very small the value of σ_u can approach the unconfined strength, q_u, of intact samples of the rock surrounding the tunnel. Similarly if D/S is very large σ_u approaches zero and the shear

strength of the rock mass approaches the shear strength along the joints. Fig. 10 gives a straight line relationship between the ratio of σ_u/q_u and the ratio of D/S which can be used to select a value of σ_u for design if q_u and D/S are known. This relationship is based on field experience in locations where displacement and strain measurements have been made.

Elastic Properties

The method of analysis presented in Chapter 2 is also quite dependent on the "effective" modulus of elasticity, E, selected for the rock mass. A method for selecting the deformation modulus of a rock mass has been given by Deere, Hendron, Patton and Cording (1967). By this method the rock quality of the rock mass must first be assessed quantitatively in terms of the Rock Quality Designation (RQD) or the Velocity Ratio as described by Deere et al (1967). After the rock quality has been determined Fig. 11 should be entered on the abscissa using the RQD value or the square of the velocity ratio and the reduction factor E_r/E_{seis} should be determined from the dotted line shown in Fig. 11. The reduction factor is the ratio of the deformation modulus of the rock mass E_r to the dynamic value of Young's modulus E_{seis} calculated from P wave velocities measured in seismic surveys. The deformation modulus of the jointed rock mass can then be taken as the product of the reduction factor and E_{seis}.

This analysis is not very sensitive to the value of Poisson's ratio selected, which is fortunate because there is little known about the selection of a Poisson's ratio for an insitu rock mass. It is recommended that a Poisson's Ratio of about 0.3 be used.

Dilatancy

Dilatancy of the rock mass at failure is accounted for in the elasto-plastic analysis by the normality condition expressed by Eq. 24. Experimental evidence however suggests that the increases in volume resulting from the normality condition are too large compared to the behavior of real rocks. Thus the radial displacements or tangential strains computed by this theory are a conservative upper bound. On the other hand the charts developed by Newmark (1969) are based upon no volume change due to the plastic strains at failure and give radial displacements and tangential strains which are too small and should be taken as a lower bound. The charts given in Figs. 3-7 would be identical to Newmark's charts if the lines of constant Q in the elastic region are extended as straight lines into the plastic region. The proper amount of dilatancy to include in calculating strains and displacements is somewhere between that given by Newmark (1969) and the analysis given herein.

Non-uniform Stress Conditions

For real design problems the free-field principal stresses in a plane perpendicular to the tunnel axis may be σ_1 and σ_3 where σ_1 is the major principal free-field stress and σ_3 is the minor free-field principal stress. The solution for this problem cannot be obtained in closed form as was done in this paper for the case of $\sigma_1 = \sigma_3 = p_o$. Reyes (1966) has solved several problems for unlined openings in a medium with Coulomb-Navier failure properties and dilatancy, as described in this report, for non-uniform free-field stress conditions. A comparison of the uniform, free-field stress solution presented herein with the solutions presented by Reyes has shown that the maximum diametral

strain δ_r/r (max) across the tunnel in the non-uniform stress field (free-field stresses = σ_1 and σ_3) is closely approximated by the solution given herein if p_o is assumed to be equal to the major principal free-field stress, σ_1. The distortion of the tunnel from a circular shape can also be approximated within 20 % if the minimum diametral strain δ_r/r (min) is taken as

$$\frac{\delta_r}{r \text{ (min)}} = \frac{\sigma_3}{\sigma_1} \frac{\delta_r}{r \text{ (max)}}$$

Since the ratio of σ_3/σ_1 for protective structures problems may range from about 1/3 to 2/3 the minimum diametral strain of the tunnel may be approximated for preliminary design purposes as being about 1/3 to 2/3 the maximum diametral strain across the tunnel. Reyes solution has also shown that the maximum diametral strain of the tunnel occurs across a diameter parallel to the direction of σ_1 and the minimum diametral strain occurs across a diameter parallel to σ_3.

For preliminary design purposes it is felt that the approximations given in this section for estimating tunnel distortions for the non-uniform stress field are at least as accurate as the initial assumption of the ratio of σ_3/σ_1 appropriate for design problems in protective structures. If more accurate estimates are desired, then time consuming and expensive finite element calculations similar to those used by Reyes (1965) must be performed.

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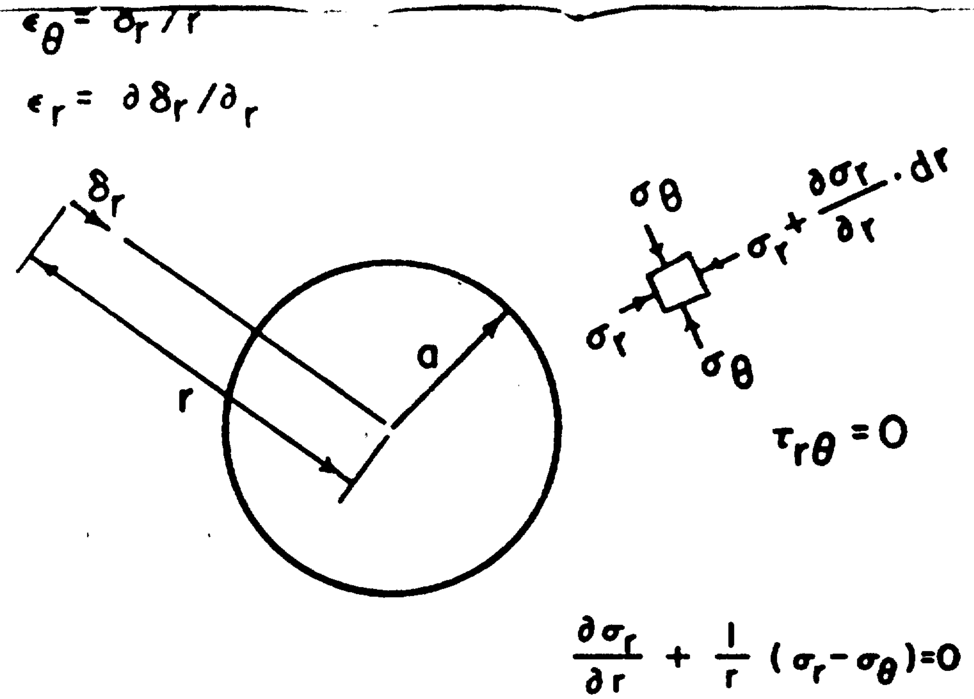


FIG. 1(a)

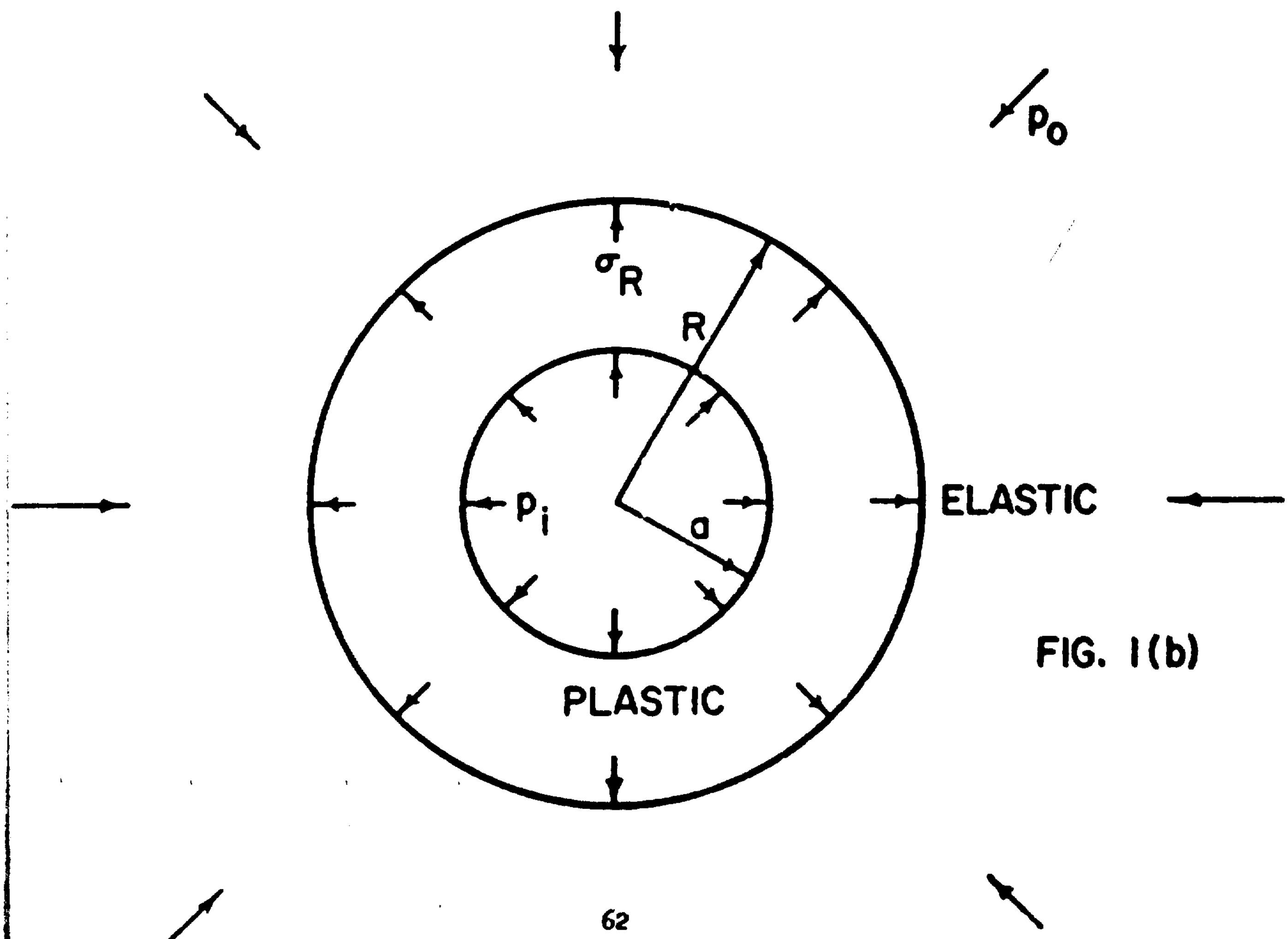


FIG. 1(b)

FIG. 1 CIRCULAR TUNNEL SUBJECTED TO AN ISOTROPIC FREE-FIELD STRE

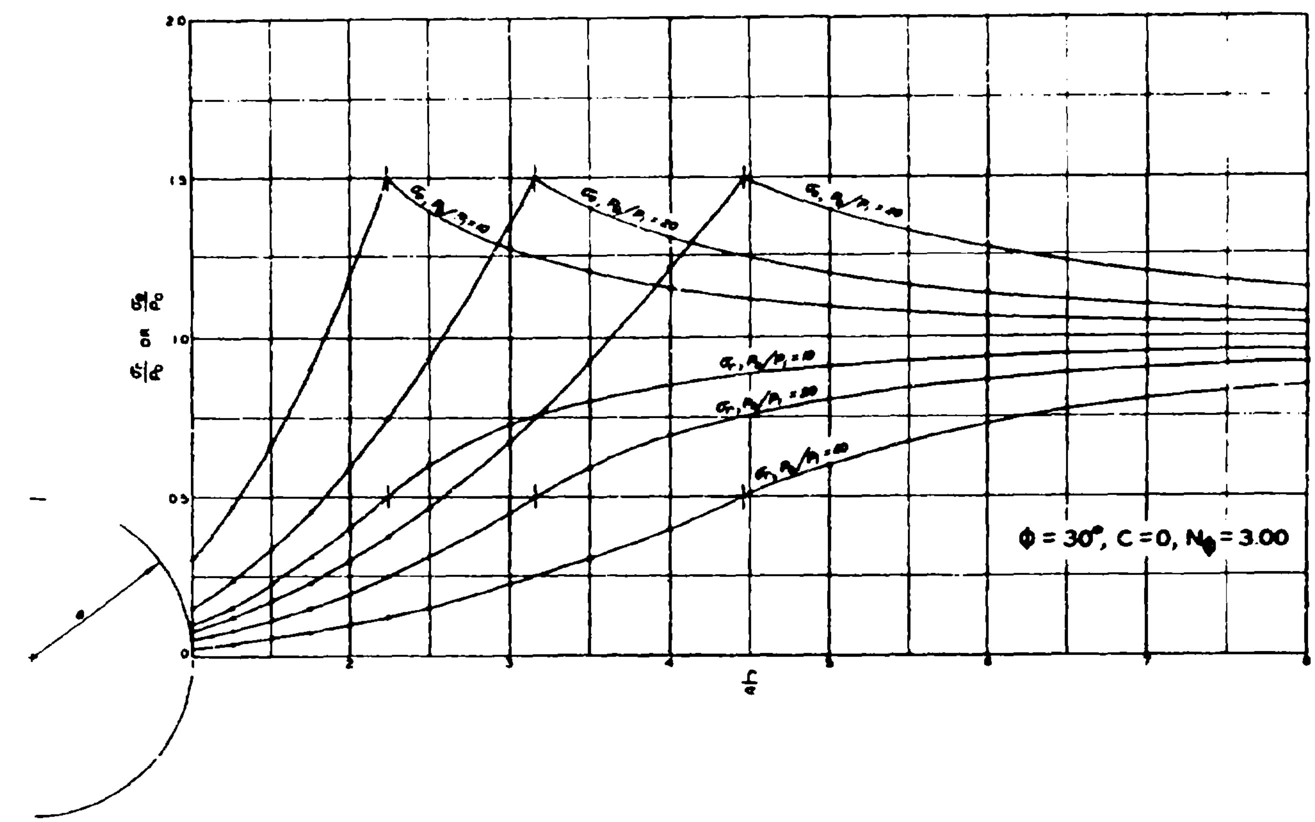


FIG. 2 ELASTIC-PLASTIC STRESS DISTRIBUTION AROUND A CIRCULAR TUNNEL IN SAND

$N_\phi = 2$

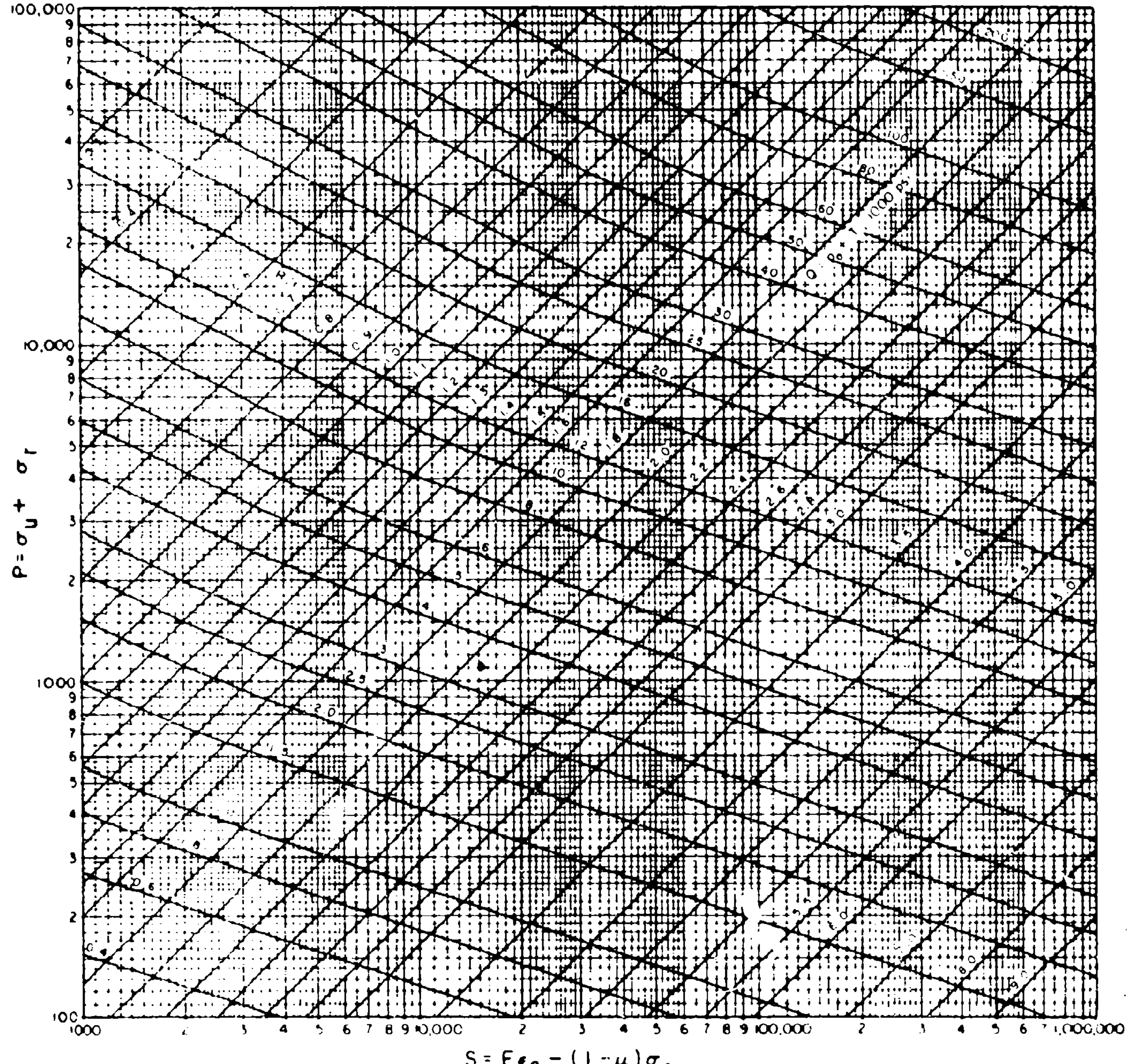


FIG 3 GRAPHICAL ILLUSTRATION OF RELATIONS AMONG P, S, AND Q FOR $N_\phi = 2$

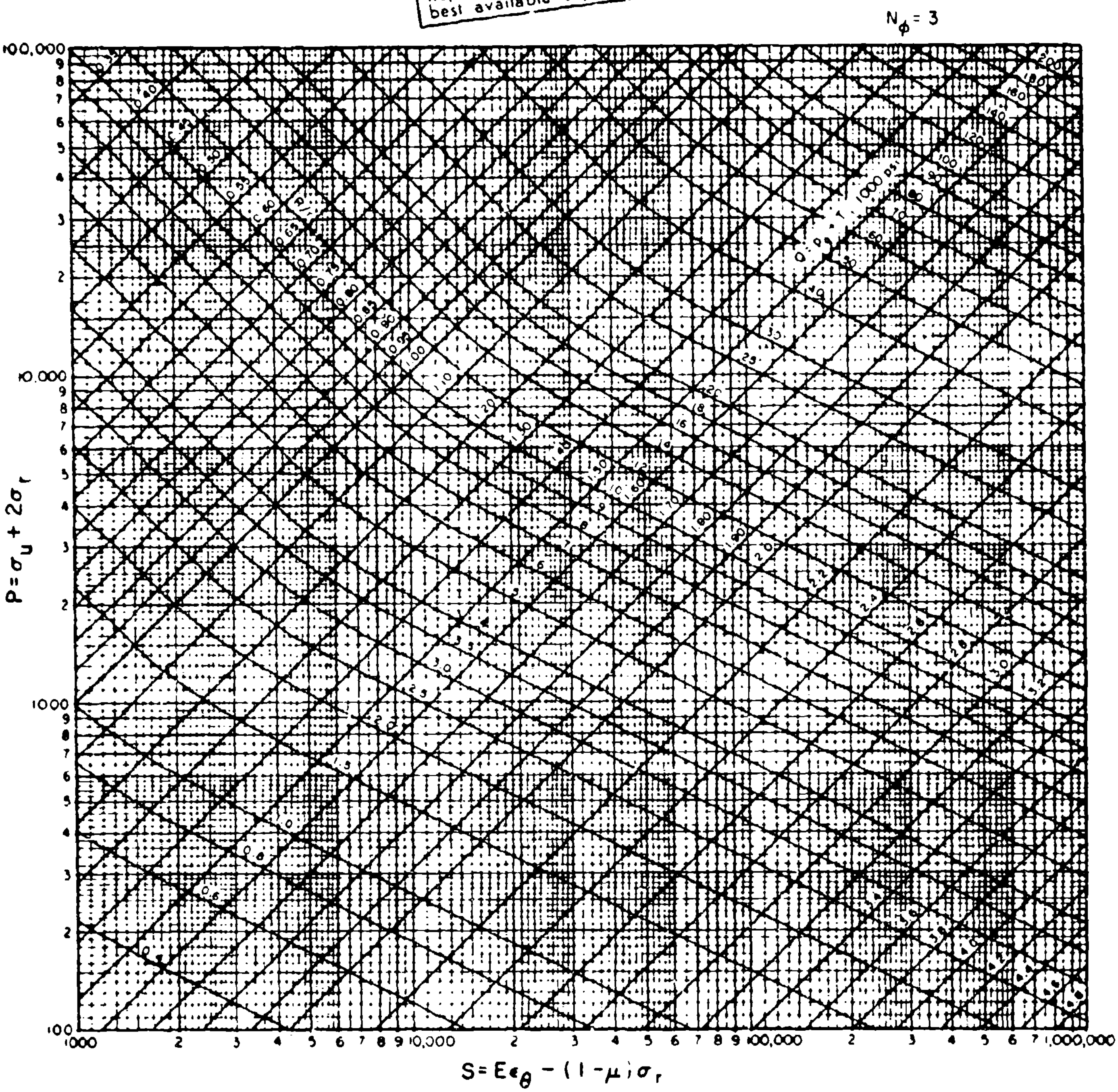


FIG. 4 GRAPHICAL ILLUSTRATION OF RELATIONS AMONG P, S, AND Q FOR $N_\phi = 3$

$N_\phi = 4$

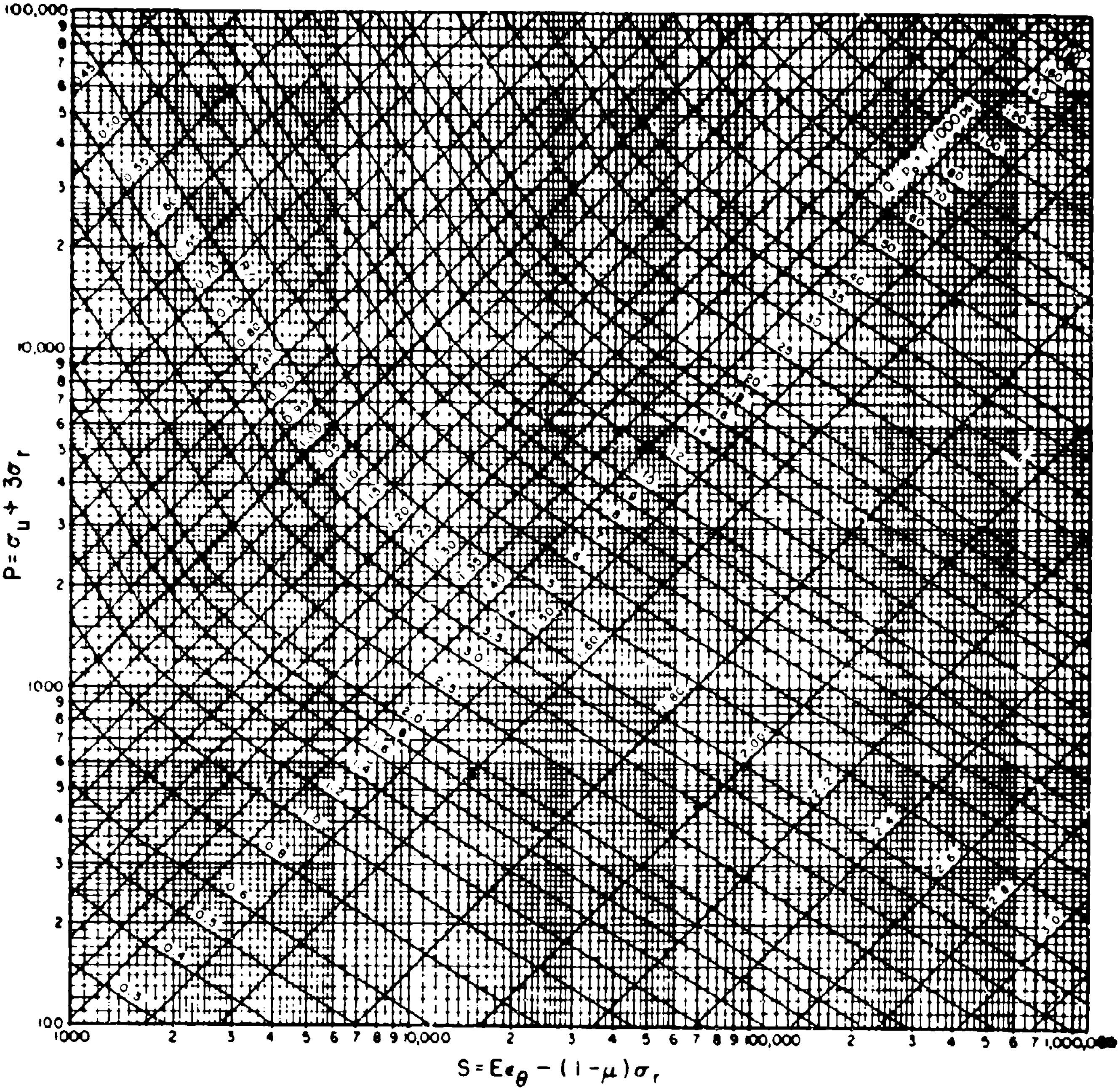


FIG. 5 GRAPHICAL ILLUSTRATION OF RELATIONS AMONG P, S, AND Q FOR N_ϕ=4

$N_\phi = 5$

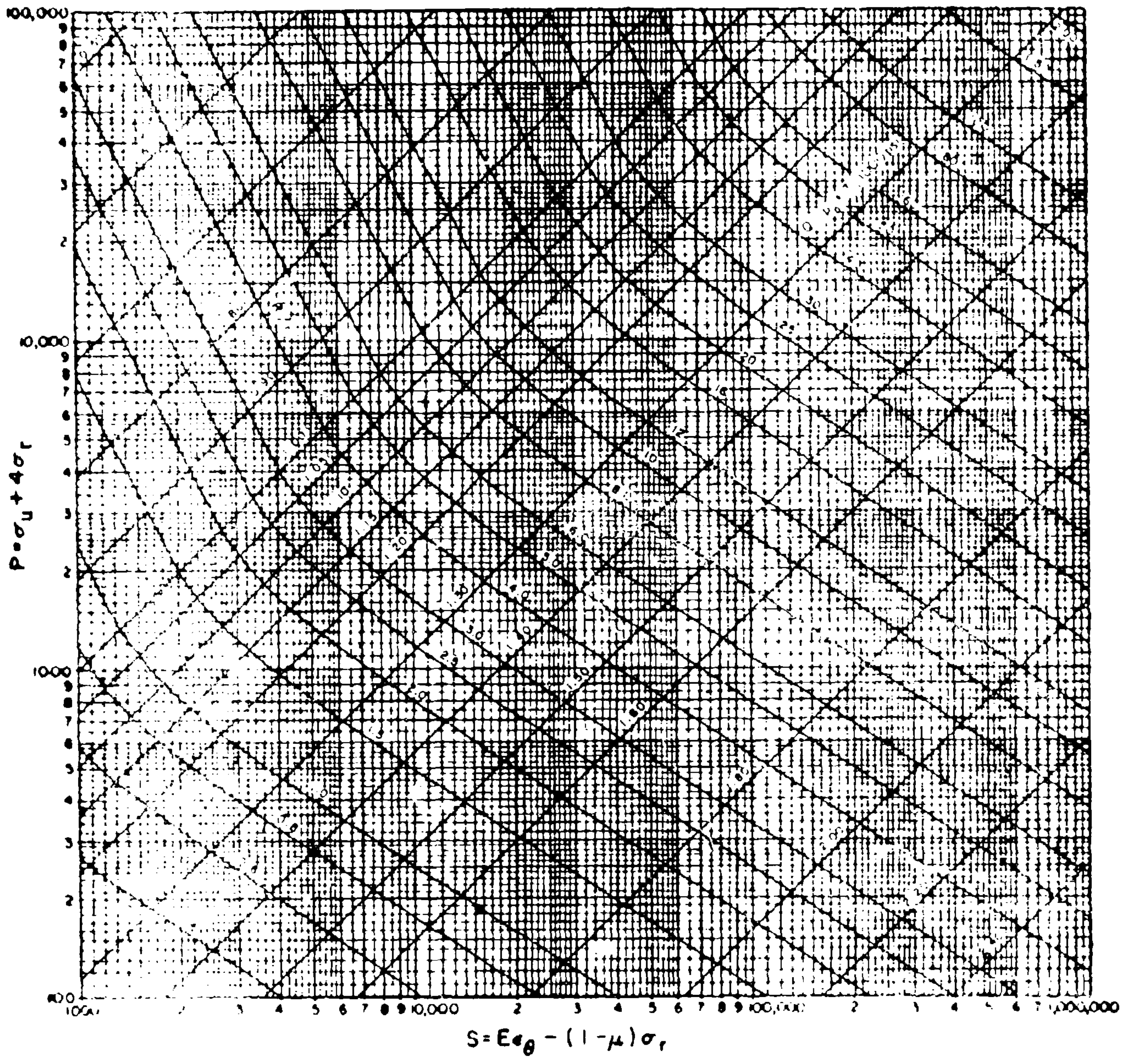


FIG. 6 ... ICAL ILLUSTRATION OF RELATIONS AMONG P, S, AND Q FOR $N_\phi = 5$

$N_\phi = 6$

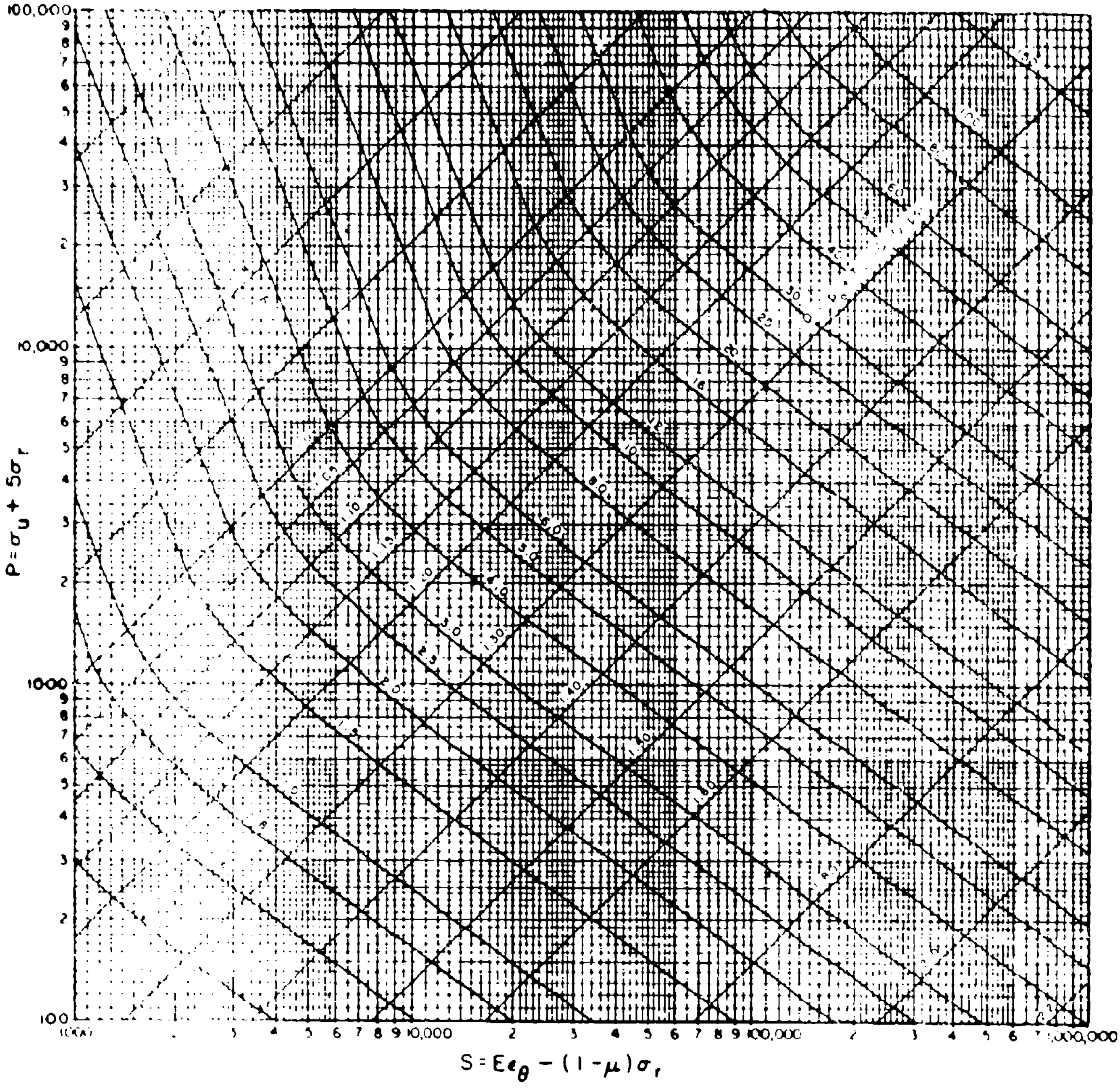


FIG. 7 ... AL ILLUSTRATION OF RELATIONS AMONG P, S, AND Q FOR $N_\phi = 6$

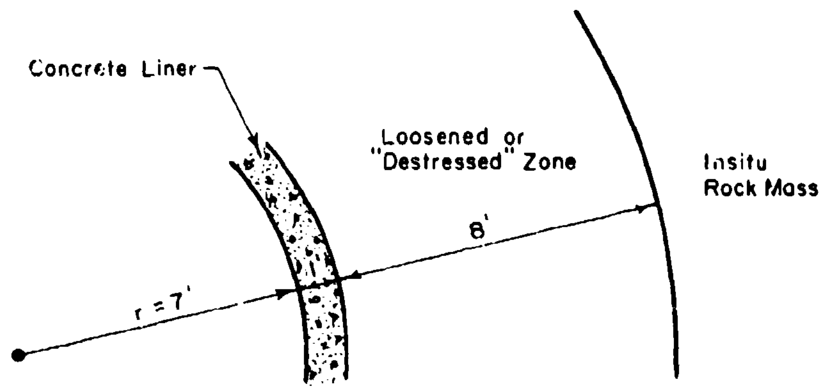
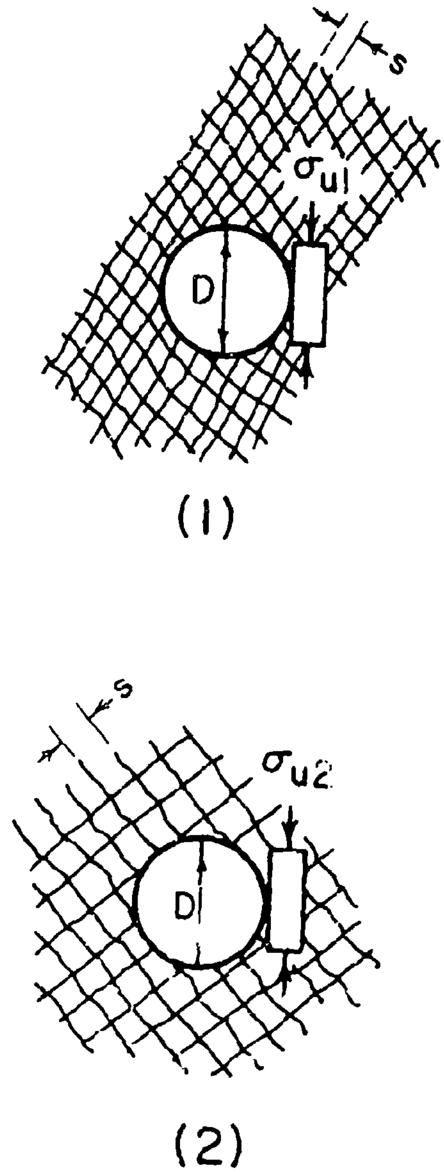


FIG. 8 GEOMETRY OF PROBLEM ASSUMED IN EXAMPLE 5



$$N_{\phi 1} \cong N_{\phi 2} = \phi_{JOINTS}$$

$$\sigma_{u1} < \sigma_{u2} = f\,(D/s)$$

FIG. 9 ILLUSTRATION OF UNCONFINED STRENGTH AS A FUNCTION OF TUNNEL DIAMETER TO JOINT SPACING

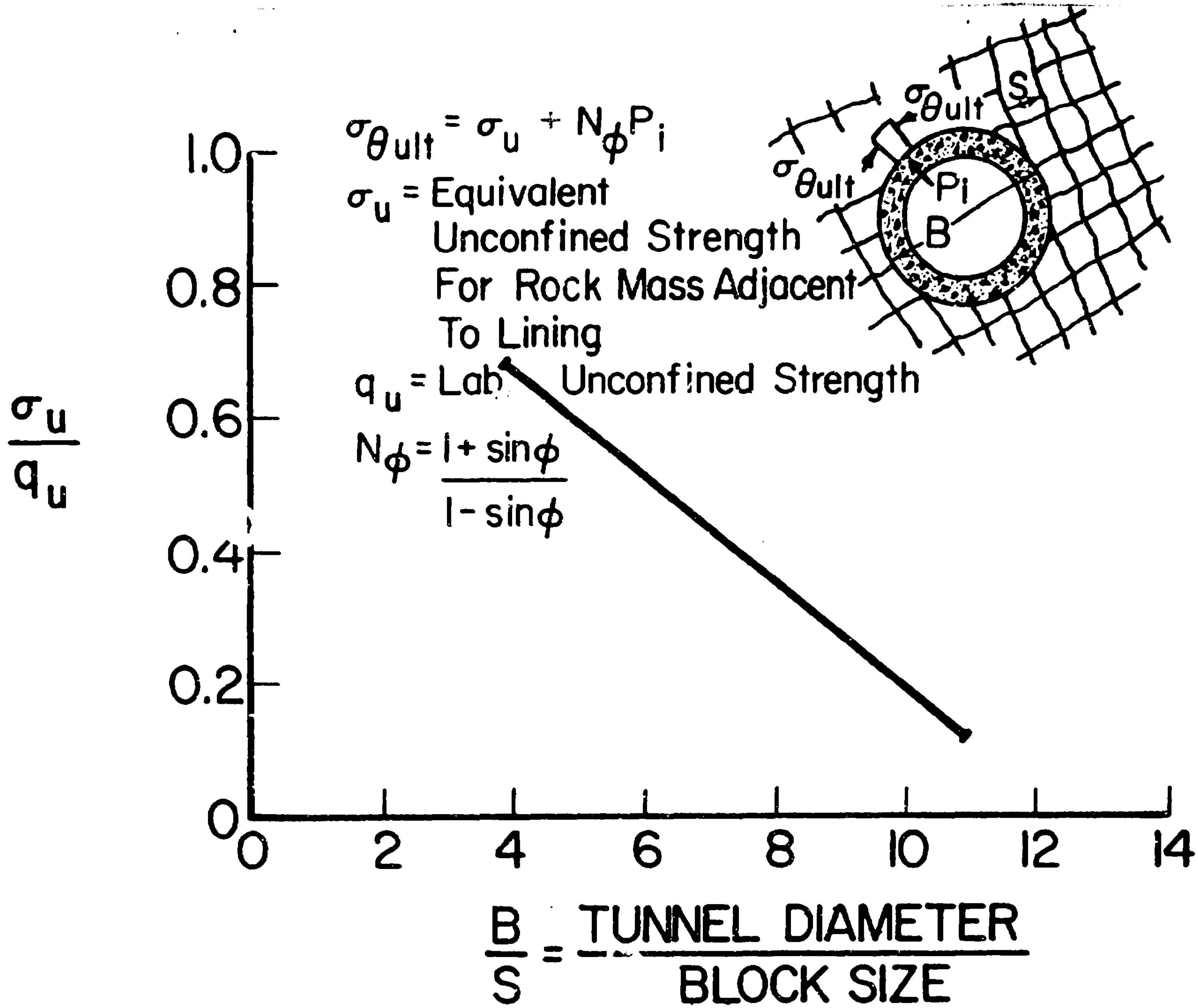


FIG. 10 RATIO OF INSITU STRENGTH TO LABORATORY STRENGTH AS A FUNCTION OF TUNNEL DIAMETER TO JOINT SPACING

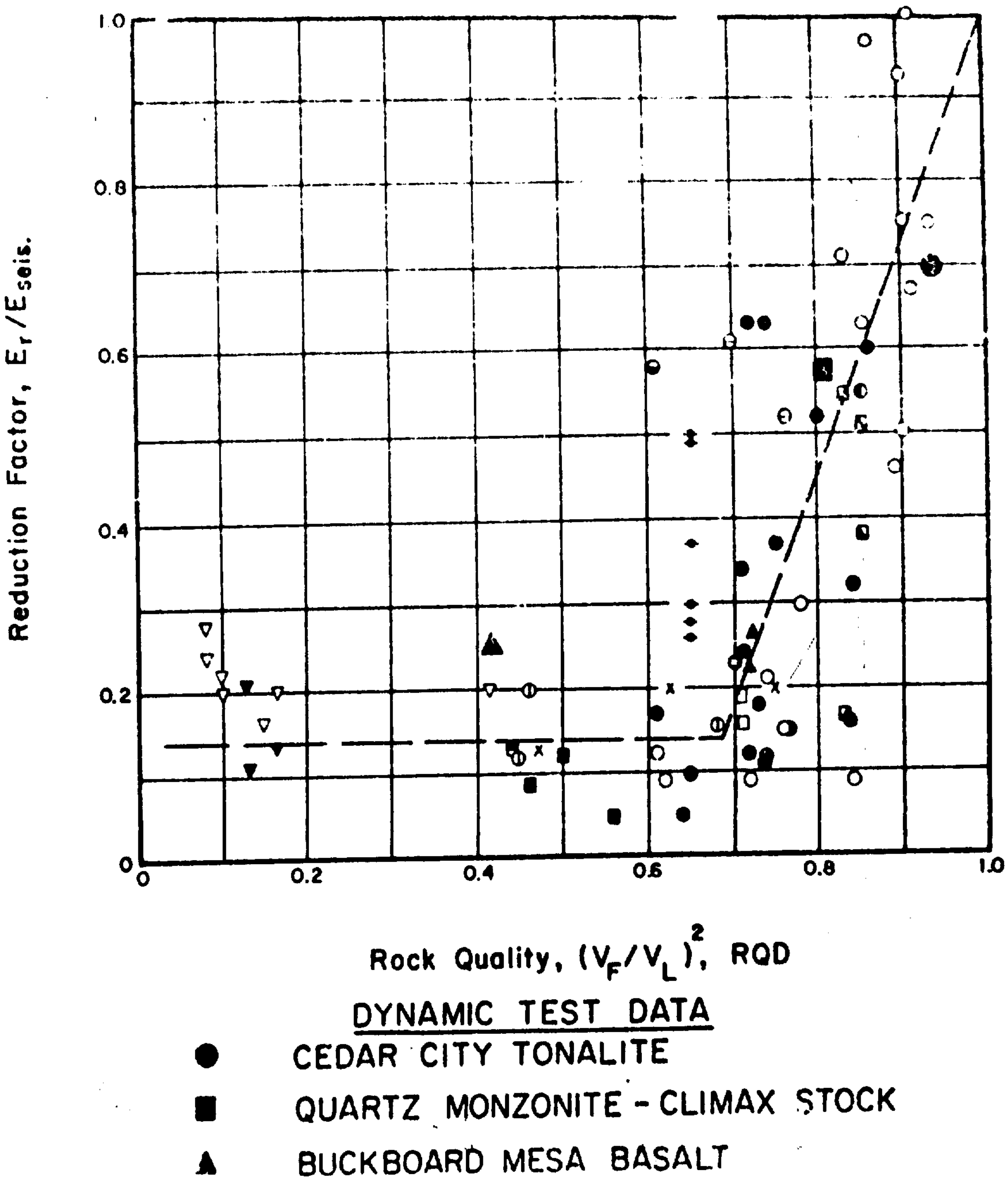


FIG. 11 RELATION BETWEEN ROCK QUALITY AND REDUCTION FACTOR